Chapter 1. Problem Solutions

1. If the speed of light were smaller than it is, would relativistic phenomena be more or less conspicuous than they are now?

【Sol】

All else being the same, including the rates of the chemical reactions that govern our brains and bodies, relativisitic phenomena would be more conspicuous if the speed of light were smaller. If we could attain the absolute speeds obtainable to us in the universe as it is, but with the speed of light being smaller, we would be able to move at speeds that would correspond to larger fractions of the speed of light, and in such instances relativistic effects would be more conspicuous.

3. An athlete has learned enough physics to know that if he measures from the earth a time interval on a moving spacecraft, what he finds will be greater than what somebody on the spacecraft would measure. He therefore proposes to set a world record for the 100-m dash by having his time taken by an observer on a moving spacecraft. Is this a good idea?

【Sol】

Even if the judges would allow it, the observers in the moving spaceship would measure a longer time, since they would see the runners being timed by clocks that appear to run slowly compared to the ship's clocks. Actually, when the effects of length contraction are included (discussed in Section 1.4 and Appendix 1), the runner's speed may be greater than, less than, or the same as that measured by an observer on the ground.

5. Two observers, A on earth and B in a spacecraft whose speed is 2.00 x 10^8 m/s, both set their watches to the same time when the ship is abreast of the earth. (a) How much time must elapse by A's reckoning before the watches differ by 1.00 s? (b) To A, B's watch seems to run slow. To B, does A's watch seem to run fast, run slow, or keep the same time as his own watch?

【Sol】

Note that the nonrelativistic approximation is not valid, as $v/c = 2/3$.

(a) See Example 1.1. In Equation (1.3), with t representing both the time measured by A and the time as measured in A's frame for the clock in B's frame to advance by to, we need

$$t - t_0 = t\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) = t\left(1 - \sqrt{1 - \left(\frac{2}{3}\right)^2}\right) = t \times 0.255 = 1.00\,\text{s}$$

from which t = 3.93 s.

(b) A moving clock always seems to run slower. In this problem, the time t is the time that observer A measures as the time that B's clock takes to record a time change of t_o.

7. How fast must a spacecraft travel relative to the earth for each day on the spacecraft to correspond to 2 *d* on the earth?

【Sol】

From Equation (1.3), for the time t on the earth to correspond to twice the time t_0 elapsed on the ship' s clock,

$$\sqrt{1-\frac{v^2}{c^2}}=\frac{1}{2}, \quad \text{so} \quad v=\frac{\sqrt{3}}{2}c=2.60\times10^8 \text{ m/s},$$

relating three significant figures.

9. A certain particle has a lifetime of 1.00 x10^{-7} s when measured at rest. How far does it go before decaying if its speed is 0.99c when it is created?

【Sol】

The lifetime of the particle is t_0, and the distance the particle will travel is, from Equation (1.3),

$$vt=\frac{vt_0}{\sqrt{1-v^2/c^2}}=\frac{(0.99)(3.0\times10^8 \text{ m/s})(1.00\times10^{-7} \text{ s})}{\sqrt{1-(0.99)^2}}=210 \text{ m}$$

to two significant figures.

11. A galaxy in the constellation Ursa Major is receding from the earth at 15,000 km/s. If one of the characteristic wavelengths of the light the galaxy emits is 550 nm, what is the corresponding wavelength measured by astronomers on the earth?

【Sol】

See Example 1.3; for the intermediate calculations, note that

$$l = \frac{c}{n} = \frac{c}{n_o}\frac{n_o}{n} = l_o\sqrt{\frac{1-v/c}{1+v/c}},$$

where the sign convention for v is that of Equation (1.8), which v positive for an approaching source and v negative for a receding source.

For this problem,

$$\frac{v}{c} = -\frac{1.50\times10^7 \text{ km/s}}{3.0\times10^8 \text{ m/s}} = -0.050,$$

so that

$$l = l_o\sqrt{\frac{1-v/c}{1+v/c}} = (550\,\text{nm})\sqrt{\frac{1+0.050}{1-0.050}} = 578\,\text{nm}$$

13. A spacecraft receding from the earth emits radio waves at a constant frequency of **10^9** Hz. If the receiver on earth can measure frequencies to the nearest hertz, at what spacecraft speed can the difference between the relativistic and classical Doppler effects be detected? For the classical effect, assume the earth is stationary.

【Sol】

This problem may be done in several ways, all of which need to use the fact that when the frequencies due to the classical and relativistic effects are found, those frequencies, while differing by 1 Hz, will both be sufficiently close to $\nu_o = 10^9$ Hz so that ν_o could be used for an approximation to either.

In Equation (1.4), we have $v = 0$ and $V = -u$, where u is the speed of the spacecraft, moving away from the earth ($V < 0$). In Equation (1.6), we have $v = u$ (or $v = -u$ in Equation (1.8)). The classical and relativistic frequencies, ν_c and ν_r respectively, are

$$n_c = \frac{n_0}{1+(u/c)}, \quad n_r = n_o\sqrt{\frac{1-(u/c)}{1+(u/c)}} = n_o\frac{\sqrt{1-(u/c)^2}}{1+(u/c)}$$

The last expression for ν_o, is motivated by the derivation of Equation (1.6), which essentially incorporates the classical result (counting the number of ticks), and allows expression of the ratio

$$\frac{n_c}{n_r} = \frac{1}{\sqrt{1-(u/c)^2}}.$$

Use of the above forms for the frequencies allows the calculation of the ratio

$$\frac{\Delta n}{n_o} = \frac{n_c - n_r}{n_o} = \frac{1-\sqrt{1-(u/c)^2}}{1+(u/c)} = \frac{1\text{ Hz}}{10^9\text{ Hz}} = 10^{-9}$$

Attempts to solve this equation exactly are not likely to be met with success, and even numerical solutions would require a higher precision than is commonly available. However, recognizing that the numerator $1-\sqrt{1-(u/c)^2}$ is of the form that can be approximated using the methods outlined at the beginning of this chapter, we can use $1-\sqrt{1-(u/c)^2} \approx (1/2)(u/c)^2$. The denominator will be indistinguishable from 1 at low speed, with the result

$$\frac{1}{2}\frac{u^2}{c^2} = 10^{-9},$$

which is solved for

$$u = \sqrt{2\times 10^{-9}}c = 1.34\times 10^4 \text{ m/s} = 13.4\text{km/s}$$

15. If the angle between the direction of motion of a light source of frequency v_o and the direction from it to an observer is 0, the frequency v the observer finds is given by

$$n = n_o \frac{\sqrt{1 - v^2/c^2}}{1 - (v/c)\cos q}$$

where v is the relative speed of the source. Show that this formula includes Eqs. (1.5) to (1.7) as special cases.

【Sol】

The transverse Doppler effect corresponds to a direction of motion of the light source that is perpendicular to the direction from it to the observer; the angle q = ±π/2 (or ±90°), so cos q = 0, and $n = n_o\sqrt{1 - v^2/c^2}$, which is Equation (1.5).

For a receding source, q = π (or 180°), and cos q = 1. The given expression becomes

$$n = n_o \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} = n_o\sqrt{\frac{1 - v/c}{1 + v/c}},$$

which is Equation (1.8).

For an approaching source, q = 0, cos q = 1, and the given expression becomes

$$n = n_o \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} = n_o\sqrt{\frac{1 + v/c}{1 - v/c}},$$

which is Equation (1.8).

17. An astronaut whose height on the earth is exactly 6 ft is lying parallel to the axis of a spacecraft moving at 0.90*c* relative to the earth. What is his height as measured by an observer in the same spacecraft? By an observer on the earth?

【Sol】

The astronaut's proper length (height) is 6 ft, and this is what any observer in the spacecraft will measure. From Equation (1.9), an observer on the earth would measure

$$L = L_o\sqrt{1 - v^2/c^2} = (6\ \text{ft})\sqrt{1-(0.90)^2} = 2.6\ \text{ft}$$

19. How much time does a meter stick moving at 0.100*c* relative to an observer take to pass the observer? The meter stick is parallel to its direction of motion.

【Sol】

The time will be the length as measured by the observer divided by the speed, or

$$t = \frac{L}{v} = \frac{L_o\sqrt{1-v^2/c^2}}{v} = \frac{(1.00\ \text{m})\sqrt{1-(0.100)^2}}{(0.100)(3.0\times 10^8\ \text{m/s})} = 3.32\times 10^{-8}\ \text{s}$$

21. A spacecraft antenna is at an angle of 10^o relative to the axis of the spacecraft. If the spacecraft moves away from the earth at a speed of 0.70*c*, what is the angle of the antenna as seen from the earth?

【Sol】

If the antenna has a length L' as measured by an observer on the spacecraft (L' is not either L or L_O in Equation (1.9)), the projection of the antenna onto the spacecraft will have a length $L'\cos(10^o)$, and the projection onto an axis perpendicular to the spacecraft's axis will have a length $L'\sin(10^o)$. To an observer on the earth, the length in the direction of the spacecraft's axis will be contracted as described by Equation (1.9), while the length perpendicular to the spacecraft's motion will appear unchanged. The angle as seen from the earth will then be

$$\arctan\left[\frac{L'\sin(10^o)}{L'\cos(10^o)\sqrt{1-v^2/c^2}}\right] = \arctan\left[\frac{\tan(10^o)}{\sqrt{1-(0.70)^2}}\right] = 14^o.$$

The generalization of the above is that if the angle is 00 as measured by an observer on the spacecraft, an observer on the earth would measure an angle *q* given by

$$\tan q = \frac{\tan q_o}{\sqrt{1-v^2/c^2}}$$

23. A woman leaves the earth in a spacecraft that makes a round trip to the nearest star, 4 light-years distant, at a speed of 0.9*c*.

【Sol】

The age difference will be the difference in the times that each measures the round trip to take, or

$$\Delta t = 2\frac{L_o}{v}\left(1-\sqrt{1-v^2/c^2}\right) = 2\frac{4\text{ yr}}{0.9}\left(1-\sqrt{1-0.9^2}\right) = 5\text{ yr.}$$

25. All definitions are arbitrary, but some are more useful than others. What is the objection to defining linear momentum as **p** = *m***v** instead of the more complicated **p** = g *m***v**?

【Sol】

It is convenient to maintain the relationship from Newtonian mechanics, in that a force on an object changes the object's momentum; symbolically, **F** = *d***p**/*dt* should still be valid. In the absence of forces, momentum should be conserved in any inertial frame, and the conserved quantity is **p** = -γ*m***v**, not *m***v**

27. Dynamite liberates about 5.4 x 10^6 J/kg when it explodes. What fraction of its total energy content is this?

【Sol】

For a given mass *M*, the ratio of the mass liberated to the mass energy is

$$\frac{M\times(5.4\times10^6\text{ J/kg})}{M\times(3.0\times10^8\text{ m/s})^2} = 6.0\times10^{-11}.$$

29. At what speed does the kinetic energy of a particle equal its rest energy?

【Sol】

If the kinetic energy $K = E_o = mc^2$, then $E = 2mc^2$ and Equation (1.23) reduces to

$$\frac{1}{\sqrt{1-v^2/c^2}} = 2$$

($\gamma = 2$ in the notation of Section 1.7). Solving for v,

$$v = \frac{\sqrt{3}}{2}c = 2.60\times 10^8 \text{ m/s}$$

31. An electron has a kinetic energy of 0.100 MeV. Find its speed according to classical and relativistic mechanics.

【Sol】

Classically,

$$v = \sqrt{\frac{2K}{m_e}} = \sqrt{\frac{2\times 0.200\,\text{MeV}\times 1.60\times 10^{-19}\text{ J/eV}}{9.11\times 10^{-31}\text{ kg}}} = 1.88\times 10^8 \text{ m/s.}$$

Relativistically, solving Equation (1.23) for v as a function of K,

$$v = c\sqrt{1-\left(\frac{m_e c^2}{E}\right)^2} = c\sqrt{1-\left(\frac{m_e c^2}{m_e c^2 + K}\right)^2} = c\sqrt{1-\left(\frac{1}{1+K/(m_e c^2)}\right)^2}.$$

With $K/(m_e c^2)$ = (0.100 MeV)/(0.511 MeV) = 0.100/0.511,

$$v = 3.0\times10^8 \text{ m/s}\times\sqrt{1-\left(\frac{1}{1+(0.100)/(0.511)}\right)^2} = 1.64\times10^8 \text{ m/s}.$$

The two speeds are comparable, but not the same; for larger values of the ratio of the kinetic and rest energies, larger discrepancies would be found.

33. A particle has a kinetic energy 20 times its rest energy. Find the speed of the particle in terms of *c*.

【Sol】

Using Equation (1.22) in Equation (1.23) and solving for v/c,

$$\frac{v}{c} = \sqrt{1-\left(\frac{E_o}{E}\right)^2}$$

With $E = 21E_o$, that is, $E = E_o + 20E_o$,

$$v = c\sqrt{1-\left(\frac{1}{21}\right)^2} = 0.9989c.$$

35. How much work (in MeV) must be done to increase the speed of an electron from 1.2 x 10^8 m/s to 2.4 X 10^8 m/s?

【Sol】

The difference in energies will be, from Equation (1.23),

$$m_e c^2 \left[\frac{1}{\sqrt{1 - v_2^2 / c^2}} - \frac{1}{\sqrt{1 - v_1^2 / c^2}} \right]$$

$$= (0.511\text{MeV}) \left[\frac{1}{\sqrt{1 - (2.4/3.0)^2}} - \frac{1}{\sqrt{1 - (1.2/3.0)^2}} \right] = 0.294 \text{ MeV}$$

37. Prove that ½$\gamma m v^2$**,** does not equal the kinetic energy of a particle moving at relativistic speeds.

【Sol】

Using the expression in Equation (1.20) for the kinetic energy, the ratio of the two quantities is

$$\frac{\frac{1}{2} g m v^2}{K} = \frac{1}{2} \frac{v^2}{c^2} \left(\frac{g}{g - 1} \right) = \frac{1}{2} \frac{v^2}{c^2} \left[\frac{1}{1 - \sqrt{1 - v^2 / c^2}} \right].$$

39. An alternative derivation of the mass-energy formula $E_O = mc^2$, also given by Einstein, is based on the principle that the location of the center of mass (CM) of an isolated system cannot be changed by any process that occurs inside the system. Figure 1.27 shows a rigid box of length L that rests on a frictionless surface; the mass M of the box is equally divided between its two ends. A burst of electromagnetic radiation of energy E_o is emitted by one end of the box. According to classical physics, the radiation has the momentum $p = E_o/c$, and when it is emitted, the box recoils with the speed $v \approx E_0 1Mc$ so that the total momentum of the system remains zero. After a time $t \approx L/c$ the radiation reaches the other end of the box and is absorbed there, which brings the box to a stop after having moved the distance S. If the CM of the box is to remain in its original place, the radiation must have transferred mass from one end to the other. Show that this amount of mass is $m = E_O 1c^2$.

【Sol】

Measured from the original center of the box, so that the original position of the center of mass is 0, the final position of the center of mass is

$$\left(\frac{M}{2} - m\right)\left(\frac{L}{2} + S\right) - \left(\frac{M}{2} + m\right)\left(\frac{L}{2} - S\right) = 0.$$

Expanding the products and canceling similar terms $[(M/2)(L/2),\ mS]$, the result $MS = mL$ is obtained. The distance 5 is the product vt, where, as shown in the problem statement, $v \approx E/Mc$ (approximate in the nonrelativistic limit $M >> Elc^2$) and $t \approx L/c$. Then,

$$m = \frac{MS}{L} = \frac{M}{L}\frac{E}{Mc}\frac{L}{c} = \frac{E}{c^2}.$$

41. In its own frame of reference, a proton takes 5 min to cross the Milky Way galaxy, which is about 10^5 light-years in diameter. (a) What is the approximate energy of the proton in electronvolts?. (b) About how long would the proton take to cross the galaxy as measured by an observer in the galaxy's reference frame?

【Sol】

To cross the galaxy in a matter of minutes, the proton must be highly relativistic, with $v \approx c$ (but $v < c$, of course). The energy of the proton will be $E = E_o\gamma$, where E_O is the proton's rest energy and $g = 1/\sqrt{1-v^2/c^2}$. However, γ, from Equation (1.9), is the same as the ratio L_O/L, where L is the diameter of the galaxy in the proton's frame of reference, and for the highly-relativistic proton $L » ct$, where t is the time in the proton's frame that it takes to cross the galaxy. Combining,

$$E = E_o g = E_o \frac{L_o}{L} \approx E_o \frac{L_o}{ct} \approx (10^9 \text{ eV}) \frac{10^5 \text{ ly}}{c(300 \text{ s})} \times (3\times10^7 \text{ s/yr}) = 10^{19} \text{ eV}$$

43. Find the momentum (in MeV/c) of an electron whose speed is 0.600c.

【Sol】

Taking magnitudes in Equation (1.16),

$$p = \frac{m_e v}{\sqrt{1-v^2/c^2}} = \frac{(0.511 \text{ MeV}/c^2)(0.600c)}{\sqrt{1-(0.600)^2}} = 0.383 \text{ MeV}/c$$

45. Find the momentum of an electron whose kinetic energy equals its rest energy of 511 keV

【Sol】

When the kinetic energy of an electron is equal to its rest energy, the total energy is twice the rest energy, and Equation (1.24) becomes

$$4m_e^4c^4 = m_e^4c^4 + p^2c^2, \quad \text{or} \quad p = \sqrt{3}(m_ec^2)/c = \sqrt{3}(511\ \text{keV}/c) = 1.94\ \text{GeV}/c$$

The result of Problem 1-29 could be used directly; $\gamma = 2$, $v = (\sqrt{3}/2)c$, and Equation (1.17) gives $p = \sqrt{3}m_ec$, as above.

47. Find the speed and momentum (in GeV/*c*) of a proton whose total energy is 3.500 GeV

【Sol】

Solving Equation (1.23) for the speed v in terms of the rest energy E_O and the total energy E,

$$v = c\sqrt{1-(E_o/E)} = c\sqrt{1-(0.938/3.500)^2} = 0.963c$$

numerically 2.888 x 10^8 m/s. (The result of Problem 1-32 does not give an answer accurate to three significant figures.) The value of the speed may be substituted into Equation (1.16) (or the result of Problem 1-46), or Equation (1.24) may be solved for the magnitude of the momentum,

$$p = \sqrt{(E/c)^2-(E_o/c)^2} = \sqrt{(3.500\ \text{GeV}/c)^2-(0.938\ \text{GeV}/c)^2} = 3.37\ \text{GeV}/c$$

49. A particle has a kinetic energy of 62 MeV and a momentum of 335 MeV/c. Find its mass (in MeV/c^2) and speed (as a fraction of c).

【Sol】

From $E = mc^2 + K$ and Equation (1.24),

$$\left(mc^2 + K\right)^2 = m^2c^4 + p^2c^2$$

Expanding the binomial, cancelling the m^2c^4 term, and solving for m,

$$m = \frac{(pc)^2 - K^2}{2c^2K} = \frac{(335\,\text{MeV})^2 - (62\,\text{MeV})^2}{2c^2(62\ \text{MeV})} = 874\ \text{MeV}\ /\ c^2.$$

The particle's speed may be found any number of ways; a very convenient result is that of Problem 1-46, giving

$$v = c^2\frac{p}{E} = c\,\frac{pc}{mc^2 + K} = c\,\frac{335\ \text{MeV}}{874\,\text{MeV} + 62\,\text{MeV}} = 0.36c.$$

51. An observer detects two explosions, one that occurs near her at a certain time and another that occurs 2.00 ms later 100 km away. Another observer finds that the two explosions occur at the, same place. What time interval separates the explosions to the second observer?

【Sol】

The given observation that the two explosions occur at the same place to the second observer means that $x' = 0$ in Equation (1.41), and so the second observer is moving at a speed

$$v = \frac{x}{t} = \frac{1.00\times10^{5}\ \text{m}}{2.00\times10^{-3}\ \text{s}} = 5.00\times10^{7}\ \text{m/s}$$

with respect to the first observer. Inserting this into Equation (1.44),

$$t' = \frac{t - \dfrac{x^2}{tc^2}}{\sqrt{1-(x/ct)^2}} = t\,\frac{1-\dfrac{x^2}{c^2t^2}}{\sqrt{1-x^2/c^2t^2}} = t\sqrt{1-\frac{(x/t)^2}{c^2}}$$

$$= (2.00\,\text{ms})\sqrt{1-\frac{(5.00\times10^{7}\ \text{m/s})^2}{(2.998\times10^{8}\ \text{m/s})^2}} = 1.97\,\text{ms}.$$

(For this calculation, the approximation $\sqrt{1-(x/ct)^2} \approx 1-(x^2/2c^2t^2)$ is valid to three significant figures.) An equally valid method, and a good cheek, is to note that when the relative speed of the observers (5.00×10^7 m/s) has been determined, the time interval that the second observer measures should be that given by Equation (1.3) (but be careful of which time it t, which is to). Algebraically and numerically, the different methods give the same result.

53. A spacecraft moving in the $+x$ direction receives a light signal from a source in the xy plane. In the reference frame of the fixed stars, the speed of the spacecraft is v and the signal arrives at an angle q to the axis of the spacecraft. (a) With the help of the Lorentz transformation find the angle q' at which the signal arrives in the reference frame of the spacecraft. (b) What would you conclude from this result about the view of the stars from a porthole on the side of the spacecraft?

【Sol】

(a) A convenient choice for the origins of both the unprimed and primed coordinate systems is the point, in both space and time, where the ship receives the signal. Then, in the unprimed frame (given here as the frame of the fixed stars, one of which may be the source), the signal was sent at a time $t = -r/c$, where r is the distance from the source to the place where the ship receives the signal, and the minus sign merely indicates that the signal was sent before it was received.

Take the direction of the ship's motion (assumed parallel to its axis) to be the positive x-direction, so that in the frame of the fixed stars (the unprimed frame), the signal arrives at an angle 0 with respect to the positive x-direction. In the unprimed frame, $x = r\cos q$ and $y = r\sin q$. From Equation (1.41),

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{r\cos q - (-r/c)}{\sqrt{1 - v^2/c^2}} = r\,\frac{\cos q + (v/c)}{\sqrt{1 - v^2/c^2}},$$

and $y' = y = r\sin q$. Then,

$$\tan q' = \frac{y'}{x'} = \frac{\sin q}{(\cos q + (v/c))/\sqrt{1-v^2/c^2}}, \quad \text{and} \quad q' = \arctan\left[\frac{\sin q\sqrt{1-v^2/c^2}}{\cos q + (v/c)}\right].$$

(b) From the form of the result of part (a), it can be seen that the numerator of the term in square brackets is less than sin$q$, and the denominator is greater than cos$q$, and so tan q and $q' < q$ when $v \neq 0$. Looking out of a porthole, the sources, including the stars, will appear to be in the directions close to the direction of the ship' s motion than they would for a ship with $v = 0$. As $v \rightarrow c$, $q' \rightarrow 0$, and all stars appear to be almost on the ship' s axis(farther forward in the field of view).

55. A man on the moon sees two spacecraft, *A* and *B*, coming toward him from opposite directions at the respective speeds of 0.800*c* and 0.900*c*. (a) What does a man on *A* measure for the speed with which he is approaching the moon? For the speed with which he is approaching *B*? (b) What does a man on *B* measure for the speed with which he is approaching the moon? For the speed with which he is approaching *A* ?

【Sol】

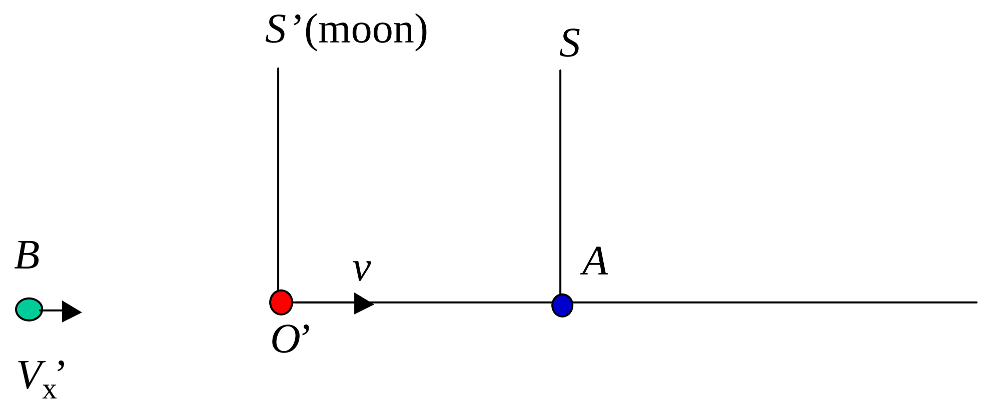
(a) If the man on the moon sees *A* approaching with speed $v = 0.800\ c$, then the observer on *A* will see the man in the moon approaching with speed $v = 0.800c$. The relative velocities will have opposite directions, but the relative speeds will be the same. The speed with which *B* is seen to approach *A*, to an observer in *A*, is then

$$V_x = \frac{V'_x + v}{1 + vV'_x/c^2} = \frac{0.800 + 0.900}{1 + (0.800)(0.900)}c = 0.988c.$$

(b) Similarly, the observer on *B* will see the man on the moon approaching with speed 0.900 *c*, and the apparent speed of *A*, to an observer on *B*, will be

$$\frac{0.900+0.800}{1+(0.900)(0.800)}c = 0.988c.$$

(Note that Equation (1.49) is unchanged if V_x' and v are interchanged.)



1. If Planck's constant were smaller than it is, would quantum phenomena be more or less conspicuous than they are now?

【Sol】

Planck's constant gives a measure of the energy at which quantum effects are observed. If Planck's constant had a smaller value, while all other physical quantities, such as the speed of light, remained the same, quantum effects would be seen for phenomena that occur at higher frequencies or shorter wavelengths. That is, quantum phenomena would be less conspicuous than they are now.

3. Is it correct to say that the maximum photoelectron energy KE_{max} is proportional to the frequency n of the incident light? If not, what would a correct statement of the relationship between KE_{max} and n be?

【Sol】

No: the relation is given in Equation (2.8) and Equation (2.9),

$$KE_{max} = hn - f = h(n - n_o),$$

So that while KE_{max} is a linear function of the frequency n of the incident light, KE_{max} is not proportional to the frequency.

5. Find the energy of a 700-nm photon.

【Sol】

From Equation (2.11),

$$E = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{700\times10^{-9}\ \text{m}} = 1.77\ \text{eV}.$$

Or, in terms of joules,

$$E = \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.0\times10^{8}\ \text{m/s})}{700\times10^{-9}\ \text{m}} = 2.84\times10^{-19}\ \text{J}$$

7. A 1.00-kW radio transmitter operates at a frequency of 880 kHz. How many photons per second does it emit?

【Sol】

The number of photons per unit time is the total energy per unit time(the power) divided by the energy per photon, or

$$\frac{P}{E} = \frac{P}{hn} = \frac{1.00\times10^{3}\ \text{J/s}}{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(880\times10^{3}\ \text{Hz})} = 1.72\times10^{30}\ \text{photons/s}.$$

9. Light from the sun arrives at the earth, an average of 1.5 x 10^{11} m away, at the rate of 1.4 **x** 10^3 W/m^2 of area perpendicular to the direction of the light. Assume that sunlight is monochromatic with a frequency of 5.0 x 10^{14} Hz. (a) How many photons fall per second on each square meter of the earth's surface directly facing the sun? (b) What is the power output of the sun, and how many photons per second does it emit? (c) How many photons per cubic meter are there near the earth?

【Sol】

(a) The number of photons per unit time per unit are will be the energy per unit time per unit area (the power per unit area, *P*/*A*), divided by the energy per photon, or

$$\frac{P\ /\ A}{hn} = \frac{1.4\times10^{3}\ \text{W/m}^{2}}{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(5.0\times10^{14}\ \text{Hz})} = 4.2\times10^{21}\ \text{photons/(s}\cdot\text{m}^{2}).$$

(b) With the reasonable assumption that the sun radiates uniformly in all directions, all points at the same distance from the sun should have the same flux of energy, even if there is no surface to absorb the energy. The total power is then,

$$(P\ /\ A)4pR_{E-S}^{2} = (1.4\times10^{3}\ \text{W/m}^{2})4p(1.5\times10^{11}\ \text{m})^{2} = 4.0\times10^{26}\ \text{W},$$

where $R_{E\text{-}S}$ is the mean Earth-Sun distance, commonly abbreviated as “1 AU,” for “astronomical unit.” The number of photons emitted per second is this power divided by the energy per photon, or

$$\frac{4.0\times10^{26}\ \text{J/s}}{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(5.0\times10^{14}\ \text{Hz})} = 1.2\times10^{45}\ \text{photons/s}.$$

(c) The photons are all moving at the same speed c, and in the same direction (spreading is not significant on the scale of the earth), and so the number of photons per unit time per unit area is the product of the number per unit volume and the speed. Using the result from part (a),

$$\frac{4.2\times10^{21}\ \text{photons/(s}\cdot\text{m}^2)}{3.0\times10^{8}\ \text{m/s}} = 1.4\times10^{13}\ \text{photons/m}^3.$$

11. The maximum wavelength for photoelectric emission in tungsten is 230 nm. What wavelength of light must be used in order for electrons with a maximum energy of 1.5 eV to be ejected?

【Sol】

Expressing Equation (2.9) in terms of $l = c/n$ and $l_0 = c/n_0$, and performing the needed algebraic manipulations,

$$l = \frac{hc}{(hc\ /\ l_o) + K_{\max}} = l_0\left[1 + \frac{K_{\max} l_o}{hc}\right]^{-1}$$

$$= (230\ \text{nm})\left[1 + \frac{(1.5\,\text{eV})(230\times10^{-9}\ \text{m})}{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}\right]^{-1} = 180\ \text{nm}.$$

13. What is the maximum wavelength of light that will cause photoelectrons to be emitted from sodium? What will the maximum kinetic energy of the photoelectrons be if 200-nm light falls on a sodium surface?

【Sol】

The maximum wavelength would correspond to the least energy that would allow an electron to be emitted, so the incident energy would be equal to the work function, and

$$l_{max} = \frac{hc}{f} = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{2.3\,\text{eV}} = 539\,\text{nm}$$

where the value of f for sodium is taken from Table 2.1.

From Equation (2.8),

$$K_{max} = hn - f = \frac{hc}{l} - f = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{200\times10^{-9}\ \text{m}} - 2.3\,\text{eV} = 3.9\,\text{eV}.$$

15. 1.5 mW of 400-nm light is directed at a photoelectric cell. If 0.10 percent of the incident photons produce photoelectrons, find the current in the cell.

【Sol】

Because only 0.10% of the light creates photoelectrons, the available power is $(1.0\text{x}10^{-3})(1.5\text{x}10^{-3}\text{W})$ = $1.5\text{x}10^{-6}$ W. the current will be the product of the number of photoelectrons per unit time and the electron charge, or

$$I = e\frac{P}{E} = e\frac{P}{hc/l} = e\frac{Pl}{hc} = (1e)\frac{(1.5\times10^{-6}\ \text{J/s})(400\times10^{-9}\ \text{m})}{1.24\times10^{-6}\ \text{eV}\cdot\text{m}} = 0.48\,m\text{A}$$

17. A metal surface illuminated by 8.5 x 10^{14} Hz light emits electrons whose maximum energy is 0.52 eV The same surface illuminated by 12.0 x 10^{14} Hz light emits electrons whose maximum energy is 1.97 eV From these data find Planck's constant and the work function of the surface .

【Sol】

Denoting the two energies and frequencies with subscripts 1 and 2,

$$K_{max,1} = hn_1 - f, \quad K_{max,2} = hn_2 - f.$$

Subtracting to eliminate the work function f and dividing by n_1 - n_2,

$$h = \frac{K_{max,2} - K_{max,1}}{n_2 - n_1} = \frac{19.7\,\text{eV} - 0.52\,\text{eV}}{12.0\times10^{14}\ \text{Hz} - 8.5\times10^{14}\ \text{Hz}} = 4.1\times10^{-15}\ \text{eV}\cdot\text{s}$$

to the allowed two significant figures. Keeping an extra figure gives

$$h = 4.14\times10^{-15}\ \text{eV}\cdot\text{s} = 6.64\times10^{-34}\ \text{J}\cdot\text{s}$$

The work function f may be obtained by substituting the above result into either of the above expressions relating the frequencies and the energies, yielding f = 3.0 eV to the same two significant figures, or the equations may be solved by rewriting them as

$$K_{max,1}n_2 = hn_1n_2 - fn_2, \quad K_{max,2}n_1 = hn_2n_1 - fn_1,$$

subtracting to eliminate the product hn_1n_2 and dividing by n_1 - n_2 to obtain

$$f = \frac{K_{max,2}n_1 - K_{max,1}n_2}{n_2 - n_1} = \frac{(19.7\,\text{eV})(8.5\times10^{14}\ \text{Hz}) - (0.52\,\text{eV})(12.0\times10^{14}\ \text{Hz})}{(12.0\times10^{14}\ \text{Hz} - 8.5\times10^{14}\ \text{Hz})} = 3.0\,\text{eV}$$

(This last calculation, while possibly more cumbersome than direct substitution, reflects the result of solving the system of equations using a symbolic-manipulation program; using such a program for this problem is, of course, a case of "swatting a fly with a sledgehammer".)

19. Show that it is impossible for a photon to give up all its energy and momentum to a free electron. This is the reason why the photoelectric effect can take place only when photons strike bound electrons.

【Sol】

Consider the proposed interaction in the frame of the electron initially at rest. The photon's initial momentum is $p_o = E_o/c$, and if the electron were to attain all of the photon's momentum and energy, the final momentum of the electron must be $p_e = p_o = p$, the final electron kinetic energy must be $KE = E_o = pc$, and so the final electron energy is $E_e = pc + m_e c^2$. However, for any electron we must have $E_e^2 = (pc)^2 + (m_e c^2)^2$. Equating the two expressions for E_e^2

$$E_e^2 = (pc)^2 + (m_e c^2)^2 = (pc + m_e c^2)^2 = (pc)^2 + 2(pc)(m_e c^2) + (m_e c^2)^2,$$

or

$$0 = 2(pc)(m_e c^2).$$

This is only possible if $p = 0$, in which case the photon had no initial momentum and no initial energy, and hence could not have existed.

To see the same result without using as much algebra, the electron's final kinetic energy is

$$\sqrt{p^2 c^2 + m_e^2 c^4} - m_e c^2 \neq pc$$

for nonzero p. An easier alternative is to consider the interaction in the frame where the electron is at rest after absorbing the photon. In this frame, the final energy is the rest energy of the electron, $m_e c^2$, but before the interaction, the electron would have been moving (to conserve momentum), and hence would have had more energy than after the interaction, and the photon would have had positive energy, so energy could not be conserved.

21. Electrons are accelerated in television tubes through potential differences of about 10 kV. Find the highest frequency of the electromagnetic waves emitted when these electrons strike the screen of the tube. What kind of waves are these?

【Sol】

For the highest frequency, the electrons will acquire all of their kinetic energy from the accelerating voltage, and this energy will appear as the electromagnetic radiation emitted when these electrons strike the screen. The frequency of this radiation will be

$$n = \frac{E}{h} = \frac{eV}{h} = \frac{(1e)(10\times10^{3}\ \text{V})}{4.14\times10^{-15}\ \text{eV}\cdot\text{s}} = 2.4\times10^{18}\ \text{Hz}$$

which corresponds to x-rays.

23. The distance between adjacent atomic planes in calcite ($CaCO_3$) is 0.300 nm. Find the smallest angle of Bragg scattering for 0.030-nm x-rays.

【Sol】

Solving Equation (2.13) for q with $n = 1$,

$$q = \arcsin\left(\frac{l}{2d}\right) = \arcsin\left(\frac{0.030\ \text{nm}}{2\times0.300\ \text{nm}}\right) = 2.9^{o}$$

25. What is the frequency of an x-ray photon whose momentum is 1.1 x 10^{-23} kg m/s**?**

【Sol】

From Equation (2.15),

$$n = \frac{cp}{h} = \frac{(3.0\times10^{8}\ \text{m/s})(1.1\times10^{-23}\ \text{kg}\cdot\text{m/s})}{6.63\times10^{-34}\ \text{J}\cdot\text{s}} = 5.0\times10^{18}\ \text{Hz}$$

27. In See. 2.7 the x-rays scattered by a crystal were assumed to undergo no change in wavelength. Show that this assumption is reasonable by calculating the Compton wavelength of a Na atom and comparing it with the typical x-ray wavelength of 0.1 nm.

【Sol】

Following the steps that led to Equation (2.22), but with a sodium atom instead of an electron,

$$l_{C,Na} = \frac{h}{cM_{Na}} = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{(3.0\times10^{8}\ \text{m/s})(3.82\times10^{-26}\ \text{kg})} = 5.8\times10^{-17}\ \text{m},$$

or 5.8 x 10^{-8} nm, which is much less than o.1 nm. (Here, the rest mass M_{Na} =3.82 x 10^{-26} kg was taken from Problem 2-24.)

29. A beam of x-rays is scattered by a target. At 45° from the beam direction the scattered x-rays have a wavelength of 2.2 pm. What is the wavelength of the x-rays in the direct beam?

【Sol】

Solving Equation (2.23) for l, the wavelength of the x-rays in the direct beam,

$$l = l' - l_C(1 - \cos f) = 2.2\ \text{pm} - (2.426\ \text{pm})(1 - \cos 45^o) = 1.5\ \text{pm}$$

to the given two significant figures.

31. An x-ray photon of initial frequency 3.0 x 10^{19} Hz collides with an electron and is scattered through 90°. Find its new frequency.

【Sol】

Rewriting Equation (2.23) in terms of frequencies, with $l = c/n$ and $l' = c/n'$, and with cos 90° = 0,

$$\frac{c}{n'} = \frac{c}{n} + l_C$$

and solving for n' gives

$$n' = \left[\frac{1}{n} + \frac{l_C}{c}\right]^{-1} = \left[\frac{1}{3.0\times10^{19}\ \text{Hz}} + \frac{2.43\times10^{-12}\ \text{m}}{3.0\times10^{8}\ \text{m/s}}\right]^{-1} = 2.4\times10^{19}\ \text{Hz}$$

The above method avoids the intermediate calculation of wavelengths.

33. At what scattering angle will incident 100-keV x-rays leave a target with an energy of 90 keV?

【Sol】

Solving Equation (2.23) for cos f,

$$\cos f = 1 + \frac{l}{l_C} - \frac{l'}{l_C} = 1 + \left(\frac{mc^2}{E} - \frac{mc^2}{E'}\right) = 1 + \left(\frac{511\,\text{keV}}{100\,\text{keV}} - \frac{511\,\text{keV}}{90\,\text{keV}}\right) = 0.432$$

from which f = 64° to two significant figures.

35. A photon of frequency n is scattered by an electron initially at rest. Verify that the maximum kinetic energy of the recoil electron is $KE_{max} = (2h^2 n^2/mc^2)/(1 + 2hn/mc^2)$.

【Sol】

For the electron to have the maximum recoil energy, the scattering angle must be 180^0, and Equation (2.20) becomes $mc^2\, KE_{max} = 2\,(hv)\,(hv')$, where $KE_{max} = (hv - hv')$ has been used. To simplify the algebra somewhat, consider

$$n' = n\frac{l}{l'} = \frac{n}{1 + (\Delta l\,/\,l)} = \frac{n}{1 + (2l_C\,/\,l)} = \frac{n}{1 + (2nl_C\,/\,c)},$$

where $\Delta l = 2\,l_C$ for f = 180°. With this expression,

$$KE_{max} = \frac{2(hn)(hn')}{mc^2} = \frac{2(hn)^2\,/\,(mc^2)}{1 + (2nl_C\,/\,c)}.$$

Using $l_C = h/(mc)$ (which is Equation (2.22)) gives the desired result.

37. A photon whose energy equals the rest energy of the electron undergoes a Compton collision with an electron. If the electron moves off at an angle of 40^o with the original photon direction, what is the energy of the scattered photon?

【Sol】

As presented in the text, the energy of the scattered photon is known in terms of the scattered angle, not the recoil angle of the scattering electron. Consider the expression for the recoil angle as given preceding the solution to Problem 2-25:

$$\tan q = \frac{\sin f}{(\Delta l / l) + (1 - \cos f)} = \frac{\sin f}{(l_C / l)(1 - \cos f) + (1 - \cos f)} = \frac{\sin f}{\left(1 + \frac{l_C}{l}\right)(1 - \cos f)}.$$

For the given problem, with $E = mc^2$, $l = hc/E = h/(mc) = l_C$, so the above expression reduces to

$$\tan q = \frac{\sin f}{2(1 - \cos f)}.$$

At this point, there are many ways to proceed; a numerical solution with $q = 40^o$ gives $f = 61.6^0$ to three significant figures. For an analytic solution which avoids the intermediate calculation of the scattering angle f, one method is to square both sides of the above relation and use the trigonometric identity $\sin^2 f = 1 - \cos^2 f = (1 + \cos f)(1 - \cos f)$ to obtain

$$4\tan^2 q = \frac{1 + \cos f}{1 - \cos f}$$

(the factor $1 - \cos f$ may be divided, as $\cos f = 1$, $f = 0$, represents an undeflected photon, and hence no interaction). This may be re-expressed as

$$(1-\cos f)(4\tan^2 q) = 1+\cos f = 2-(1-\cos f), \quad \text{or}$$

$$1-\cos f = \frac{2}{1+4\tan^2 q}, \quad 2-\cos f = \frac{3+4\tan^2 q}{1+4\tan^2 q}.$$

Then with $l' = l + l_C(1-\cos f) = l_C(2-\cos f)$,

$$E' = E\frac{l}{l'} = E\frac{1+4\tan^2 q}{3+4\tan^2 q} = (511\,\text{keV})\frac{1+4\tan^2(40^o)}{3+4\tan^2(40^o)} = 335\,\text{eV}$$

An equivalent but slightly more cumbersome method is to use the trigonometric identities

$$\sin f = 2\sin\frac{f}{2}\cos\frac{f}{2}, \quad 1-\cos f = 2\sin^2\frac{f}{2}$$

in the expression for $\tan q$ to obtain

$$\tan q = \frac{1}{2}\cot\frac{f}{2}, \quad f = 2\arctan\left(\frac{1}{2\tan q}\right)$$

yielding the result $q = 61.6^o$ more readily.

39. A positron collides head on with an electron and both are annihilated. Each particle had a kinetic energy of 1.00 MeV Find the wavelength of the resulting photons.

【Sol】

The energy of each photon will he the sum of one particle's rest and kinetic energies, 1.511 MeV (keeping an extra significant figure). The wavelength of each photon will be

$$l = \frac{hc}{E} = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{1.51\times10^{6}\ \text{eV}} = 8.21\times10^{-13}\ \text{m} = 0.821\,\text{pm}$$

41. Show that, regardless of its initial energy, a photon cannot undergo Compton scattering through an angle of more than 60º and still be able to produce an electron-positron pair. (Hint: Start by expressing the Compton wavelength of the electron in terms of the maximum photon wavelength needed for pair production.)

【Sol】

Following the hint,

$$l_C = \frac{h}{mc} = \frac{2hc}{2mc^2} = \frac{2hc}{E_{\min}},$$

where $E_{\min} = 2mc^2$ is the minimum photon energy needed for pair production. The scattered wave-length (a maximum) corresponding to this minimum energy is $l'_{\max} = (h/E_{\min})$, so $l_C = 2l'_{\max}$. At this point, it is possible to say that for the most energetic incoming photons, $l \sim 0$, and so $1 - \cos f = ½$ for $l' = l_C/2$, from which $\cos f = ½$ and $f = 60º$. As an alternative, the angle at which the scattered photons will have wavelength $l'_{\max}$ can m be found as a function of the incoming photon energy E; solving Equation (2.23) with $l' = l'_{\max}$)

$$\cos f = 1 - \frac{l'_{\max} - l}{l_C} = 1 - \frac{l'_{\max}}{l_C} + \frac{hc/E}{l_C} = \frac{1}{2} + \frac{mc^2}{E}.$$

This expression shows that for $E >> mc^2$, cos f = ½ and so f = 60°, but it also shows that, because cos f must always be less than 1, for pair production at any angle, E must be greater than $2mc^2$, which we know to be the case.

43. (a) Show that the thickness $x_{1/2,}$ of an absorber required to reduce the intensity of a beam of radiation by a factor of 2 is given by $x_{1/2}$ = 0.693/μ. (b) Find the absorber thickness needed to produce an intensity reduction of a factor of 10.

【Sol】

(a) The most direct way to get this result is to use Equation (2.26) with I_o/I = 2, so that

$$I = I_o e^{-mx} \Rightarrow x_{1/2} = \frac{\ln 2}{m} = \frac{0.693}{m}.$$

(b) Similarly, with I_o/I = 10,

$$x_{1/10} = \frac{\ln 10}{m} = \frac{2.30}{m}.$$

45. The linear absorption coefficient for 1-MeV gamma rays in lead is 78 m^{-1}. find the thickness of lead required to reduce by half the intensity of a beam of such gamma rays.

【Sol】

From either Equation (2.26) or Problem 2-43 above,

$$x_{1/2} = \frac{\ln 2}{m} = \frac{0.693}{78\,\mathrm{m}^{-1}} = 8.9\,\mathrm{mm}$$

47. The linear absorption coefficients for 2.0-MeV gamma rays are 4.9 m^{-1} in water and 52 in $^{-1}$ in lead. What thickness of water would give the same shielding for such gamma rays as 10 mm of lead?

【Sol】

Rather than calculating the actual intensity ratios, Equation (2.26) indicates that the ratios will be the same when the distances in water and lead are related by

$$m_{H_2O}\, x_{H_2O} = m_{Pb}\, x_{Pb}, \quad \text{or}$$

$$x_{H_2O} = x_{Pb} \frac{m_{Pb}}{m_{H_2O}} = (10 \times 10^{-3}\ \mathrm{m}) \frac{52\ \mathrm{m}^{-1}}{4.9\ \mathrm{m}^{-1}} = 0.106\,\mathrm{m}$$

or 11 cm two significant figures.

49. What thickness of copper is needed to reduce the intensity of the beam in Exercise 48 by half.

【Sol】

Either a direct application of Equation (2.26) or use of the result of Problem 2-43 gives

$$x_{1/2} = \frac{\ln 2}{4.7 \times 10^{4}\ \text{m}^{-1}} = 1.47 \times 10^{-5}\ \text{m},$$

which is 0.015 mm to two significant figures.

51. The sun's mass is 2.0 x 10^{30} kg and its radius is 7.0 x 10^{8} m. Find the approximate gravitational red shift in light of wavelength 500 nm emitted by the sun.

【Sol】

In Equation (2.29), the ratio

$$\frac{GM}{c^2 R} = \frac{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg})(2.0 \times 10^{30}\ \text{kg})}{(3.0 \times 10^{8}\ \text{m/s})^2 (7.0 \times 10^{4}\ \text{m}^{-1})} = 2.12 \times 10^{-6}$$

(keeping an extra significant figure) is so small that for an "approximate" red shift, the ratio $\Delta l / l$ will be the same as $\Delta n / n$, and

$$\Delta l = l \frac{GM}{c^2 R} = (500 \times 10^{-9}\ \text{m})(2.12 \times 10^{-6}) = 1.06 \times 10^{-12}\ \text{m} = 1.06\ \text{pm}.$$

53. As discussed in Chap. 12, certain atomic nuclei emit photons in undergoing transitions from "excited" energy states to their "ground" or normal states. These photons constitute gamma rays. When a nucleus emits a photon, it recoils in the opposite direction. (a) The $^{57}_{27}Co$ nucleus decays by *K* capture to $^{57}_{26}Fe$, which then emits a photon in losing 14.4 keV to reach its ground state. The mass of a $^{57}_{26}Fe$ atom is 9.5 x 10^{-26} kg. By how much is the photon energy reduced from the full 14.4 keV available as a result of having to share energy and momentum with the recoiling atom? (b) In certain crystals the atoms are so tightly bound that the entire crystal recoils when a gamma-ray photon is emitted, instead of the individual atom. This phenomenon is known as the **Mössbauer effect.** By how much is the photon energy reduced in this situation if the ex- cited 2576Fe nucleus is part of a 1.0-g crystal? (c) The essentially recoil-free emission of gamma rays in situations like that of b means that it is possible to construct a source of virtually mono-energetic and hence monochromatic photons. Such a source was used in the experiment described in See. 2.9. What is the original frequency and the change in frequency of a 14.4-keV gamma-ray photon after it has fallen 20 m near the earth's surface?

【Sol】

(a) The most convenient way to do this problem, for computational purposes, is to realize that the nucleus will be moving nonrelativistically after the emission of the photon, and that the energy of the photon will be very close to E_∞ = 14.4 keV, the energy that the photon would have if the nucleus had been infinitely massive. So, if the photon has an energy *E*, the recoil momentum of the nucleus is E/c, and its kinetic energy is $p^2/2M = E^2/(2Mc^2)$, here *M* is the rest mass of the nucleus. Then, conservation of energy implies

$$\frac{E^2}{2Mc^2} + E = E_\infty.$$

This is a quadratic in E, and solution might be attempted by standard methods, but to find the change in energy due to the finite mass of the nucleus, and recognizing that E will be very close to E_∞, the above relation may be expressed as

$$E_\infty - E = \frac{E^2}{2Mc^2} \approx \frac{E_\infty^2}{2Mc^2}$$

$$= \frac{(14.4\text{ keV})^2(1.60\times10^{-16}\text{ J/keV})}{2(9.5\times10^{-26}\text{ kg})(3.0\times10^8\text{ m/s})^2} = 1.9\times10^6\text{ keV} = 1.9\times10^3\text{ eV}.$$

If the approximation $E \approx E_\infty$, is not made, the resulting quadratic is

$$E^2 + 2Mc^2E - 2Mc^2E_\infty = 0,$$

which is solved for

$$E = Mc^2\left[\sqrt{1+2\frac{E_\infty}{Mc^2}} - 1\right].$$

However, the dimensionless quantity $E_\infty/(Mc^2)$ is so small that standard calculators are not able to determine the difference between E and E_∞. The square root must be expanded, using $(1 + x)^{1/2} \approx 1 + (x/2) - (x^2/8)$, and two terms must be kept to find the difference between E and E_∞. This approximation gives the previous result.

It so happens that a relativistic treatment of the recoiling nucleus gives the same numerical result, but without intermediate approximations or solution of a quadratic equation. The relativistic form expressing conservation of energy is, with $pc = E$ and before,

$$\sqrt{E^2 + (Mc^2)^2} + E = Mc^2 + E_\infty, \quad \text{or} \quad \sqrt{E^2 + (Mc^2)^2} = Mc^2 + E_\infty - E.$$

Squaring both sides, canceling E^2 and $(Mc^2)^2$, and then solving for E,

$$E = \frac{E_\infty^2 + 2Mc^2 E_\infty}{2(Mc^2 + E_\infty)} = E_\infty \left(\frac{1 + (E_\infty / (2Mc^2))}{1 + (E_\infty / (Mc^2))} \right).$$

From this form,

$$E_\infty - E = \left(\frac{E_\infty^2}{2Mc^2} \right) \frac{1}{1 + E_\infty / (Mc^2)},$$

giving the same result.

(b) For this situation, the above result applies, but the nonrelativistic approximation is by far the easiest for calculation;

$$E_\infty - E = \frac{E_\infty^2}{2Mc^2} = \frac{(14.4 \times 10^3 \text{ eV})^2 (1.6 \times 10^{-19} \text{ J/eV})}{2(1.0 \times 10^{-3} \text{ kg})(3.0 \times 10^8 \text{ m/s})^2} = 1.8 \times 10^{-25} \text{ eV}.$$

(c) The original frequency is $n = \dfrac{E_\infty}{h} = \dfrac{14.4 \times 10^3 \text{ eV}}{4.14 \times 10^{-15} \text{ eV} \cdot \text{s}} = 3.48 \times 10^{18} \text{ Hz}.$

From Equation (2.28), the change in frequency is

$$\Delta n = n' - n = \left(\frac{gH}{c^2} \right) n = \frac{(9.8 \text{ m/s}^2)(20 \text{ m})}{(3.0 \times 10^8 \text{ m/s})^2} (3.48 \times 10^{18} \text{ Hz}) = 7.6 \text{ Hz}.$$

55. The gravitational potential energy U relative to infinity of a body of mass m at a distance R from the center of a body of mass M is $U = -GmM/R$. (a) If R is the radius of the body of mass M, find the escape speed v, of the body, which is the minimum speed needed to leave it permanently. (b) Obtain a formula for the Schwarzschild radius of the body by setting $v_c = c$, the speed of light, and solving for R. (Of course, a relativistic calculation is correct here, but it is interesting to see what a classical calculation produces.)

【Sol】

(a) To leave the body of mass M permanently, the body of mass m must have enough kinetic energy so that there is no radius at which its energy is positive. That is, its total energy must be nonnegative. The escape velocity v_e is the speed (for a given radius, and assuming $M >> m$) that the body of mass m would have for a total energy of zero;

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0, \quad \text{or} \quad v_e = \sqrt{\frac{2GM}{R}}.$$

(b) Solving the above expression for R in terms of v_e,

$$R = \frac{2GM}{v_e^2},$$

and if $v_e = c$, Equation (2.30) is obtained.

1. A photon and a particle have the same wavelength. Can anything be said about how their linear momenta compare? About how the photon's energy compares with the particle's total energy? About how the photon' s energy compares with the particle's kinetic energy?

【Sol】

From Equation (3.1), any particle' s wavelength is determined by its momentum, and hence particles with the same wavelength have the same momenta. With a common momentum p, the photon' s energy is pc, and the particle' s energy is $\sqrt{(pc)^2+(mc^2)^2}$, which is necessarily greater than pc for a massive particle. The particle' s kinetic energy is

$$K = E - mc^2 = \sqrt{(pc)^2 + \left(mc^2\right)^2} - mc^2$$

For low values of p ($p \ll mc$ for a nonrelativistic massive particle), the kinetic energy is $K \approx p^2/2m$, which is necessarily less than pc. For a relativistic massive particle, $K \approx pc - mc^2$, and K is less than the photon energy. The kinetic energy of a massive particle will always be less than pc, as can be seen by using $E = (pc)^2 + (mc^2)^2$ to obtain

$$(pc)^2 - K^2 = 2Kmc^2.$$

3. Find the de Broglie wavelength of a 1.0-mg grain of sand blown by the wind at a speed of 20 m/s.

【Sol】

For this nonrelativistic case,

$$l = \frac{h}{mv} = \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{(1.0\times10^{-6}\ \text{kg})(20\ \text{m/s})} = 3.3\times10^{-29}\ \text{m};$$

quantum effects certainly would not be noticed for such an object.

5. By what percentage will a nonrelativistle calculation of the de Broglie wavelength of a 100-keV electron be in error?

【Sol】

Because the de Broglie wavelength depends only on the electron's momentum, the percentage error in the wavelength will be the same as the percentage error in the reciprocal of the momentum, with the nonrelativistic calculation giving the higher wavelength due to a lower calculated momentum. The nonrelativistic momentum is

$$p_{nr} = \sqrt{2mK} = \sqrt{2(9.1\times10^{-31}\ \text{kg})(100\times10^{3}\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})}$$
$$= 1.71\times10^{-22}\ \text{kg}\cdot\text{m}/\text{s},$$

and the relativistic momentum is

$$p_r = \frac{1}{c}\sqrt{\left(K+mc^2\right)^2-\left(mc^2\right)^2} = \sqrt{(0.100+(0.511)^2}\ \text{MeV}/c = 1.79\times10^{-22}\ \text{kg}\cdot\text{m/s},$$

keeping extra figures in the intermediate calculations. The percentage error in the computed de Broglie wavelength is then

$$\frac{(h/p_{nr})-(h/p_r)}{h/p_r}=\frac{p_r-p_{nr}}{p_{nr}}=\frac{1.79-1.71}{1.71}=4.8\ \%.$$

7. The atomic spacing in rock salt, NaCl, is 0.282 nm. Find the kinetic energy (in eV) of a neutron with a de Broglie wavelength of 0.282 nm. Is a relativistic calculation needed? Such neutrons can be used to study crystal structure.

【Sol】

A nonrelativistic calculation gives

$$K=\frac{p^2}{2m}=\frac{(hc/l)^2}{2mc^2}=\frac{(hc)^2}{2mc^2l^2}=\frac{(1.24\times10^{-6}\text{ eV}\cdot\text{m})^2}{2(939.6\times10^6\text{ eV})(0.282\times10^{-9}\text{ m})^2}=1.03\times10^{-3}\text{ eV}$$

(Note that in the above calculation, multiplication of numerator and denominator by c^2 and use of the product hc in terms of electronvolts avoided further unit conversion.) This energy is much less than the neutron's rest energy, and so the nonrelativistic calculation is completely valid.

9. Green light has a wavelength of about 550 nm. Through what potential difference must an electron be accelerated to have this wavelength?

【Sol】

A nonrelativistic calculation gives

$$K = \frac{p^2}{2m} = \frac{(hc/l)^2}{2mc^2} = \frac{(hc)^2}{2(mc^2)l^2} = \frac{(1.24\times10^{-6}\ \text{eV}\cdot\text{m})^2}{2(511\times10^3\ \text{eV})(550\times10^{-9}\ \text{m})^2} = 5.0\times10^{-6}\ \text{eV},$$

so the electron would have to be accelerated through a potential difference of 5.0 x 10^{-6} V = 5.0 μV. Note that the kinetic energy is very small compared to the electron rest energy, so the nonrelativistic calculation is valid. (In the above calculation, multiplication of numerator and denominator by c^2 and use of the product he in terms of electronvolts avoided further unit conversion.)

11. Show that if the total energy of a moving particle greatly exceeds its rest energy, its de Broglie wavelength is nearly the same as the wavelength of a photon with the same total energy.

【Sol】

If $E^2 = (pc)^2 + (mc^2)^2 >> (mc^2)^2$, then $pc >> mc^2$ and $E \approx pc$. For a photon with the same energy, $E = pc$, so the momentum of such a particle would be nearly the same as a photon with the same energy, and so the de Broglie wavelengths would be the same.

13. An electron and a proton have the same velocity Compare the wavelengths and the phase and group velocities of their de Broglie waves.

【Sol】

For massive particles of the same speed, relativistic or nonrelativistic, the momentum will be proportional to the mass, and so the de Broglie wavelength will be inversely proportional to the mass; the electron will have the longer wavelength by a factor of $(m_p/m_e) = 1838$. From Equation (3.3) the particles have the same phase velocity and from Equation (3.16) they have the same group velocity.

15. Verify the statement in the text that, if the phase velocity is the same for all wavelengths of a certain wave phenomenon (that is, there is no dispersion), the group and phase velocities are the same.

【Sol】

Suppose that the phase velocity is independent of wavelength, and hence independent of the wave number k; then, from Equation (3.3), the phase velocity $v_p = (w/k) = u$, a constant. It follows that because $w = uk$,

$$v_g = \frac{dw}{dk} = u = v_p .$$

17. The phase velocity of ocean waves is $\sqrt{gl/2p}$, where g is the acceleration of gravity. Find the group velocity of ocean waves

【Sol】

The phase velocity may be expressed in terms of the wave number $k = 2\pi/\lambda$ as

$$v_p = \frac{w}{k} = \sqrt{\frac{g}{k}}, \quad \text{or} \quad w = \sqrt{gk} \quad \text{or} \quad w^2 = gk.$$

Finding the group velocity by differentiating $w(k)$ with respect to k,

$$v_g = \frac{dw}{dk} = \frac{1}{2}\sqrt{g}\,\frac{1}{\sqrt{k}} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{w}{k} = \frac{1}{2}v_p.$$

Using implicit differentiation in the formula for $w^2(k)$,

$$2w\frac{dw}{dk} = 2wv_g = g,$$

so that $$v_g = \frac{g}{2w} = \frac{gk}{2wk} = \frac{w^2}{2wk} = \frac{w}{2k} = \frac{1}{2}v_p,$$

the same result. For those more comfortable with calculus, the dispersion relation may be expressed as

$$2\ln(w) = \ln(k) + \ln(g),$$

from which $2\dfrac{dw}{w} = \dfrac{dk}{k}$, and $v_g = \dfrac{1}{2}\dfrac{w}{k} = \dfrac{1}{2}v_p$.

19. Find the phase and group velocities of the de Broglie waves of an electron whose kinetic energy is 500 keV.

【Sol】

For a kinetic energy of 500 keV, $g = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{K+mc^2}{mc^2} = \frac{500+511}{511} = 1.978.$

Solving for v,

$$v = c\sqrt{1-(1/g)^2} = c\sqrt{1-(1/1.978)^2} = 0.863c,$$

and from Equation (3.16), $v_g = v = 0.863c$. The phase velocity is then $v_p = c^2/v_g = 1.16\,c$.

21. (a) Show that the phase velocity of the de Broglie waves of a particle of mass m and de Broglie wavelength l is given by

$$v_p = c\sqrt{1+\left(\frac{mcl}{h}\right)^2}$$

(b) Compare the phase and group velocities of an electron whose de Broglie wavelength is exactly 1 x 10^{-13} m.

【Sol】

(a) Two equivalent methods will be presented here. Both will assume the validity of Equation (3.16), in that $v_g = v$.

First: Express the wavelength x in terms of v_g,

$$l = \frac{h}{p} = \frac{h}{mv_g g} = \frac{h}{mv_g}\sqrt{1-\frac{v_g^2}{c^2}}.$$

Multiplying by mv_g, squaring and solving for v_g^2 gives

$$v_g^2 = \frac{h^2}{(l\,m)^2 + (h^2/c^2)} = c^2\left[1 + \left(\frac{m\,l\,c}{h}\right)^2\right]^{-1}.$$

Taking the square root and using Equation (3.3), $v_p = c^2/v_g$, gives the desired result.

Second: Consider the particle energy in terms of $v_p = c^2\,lv_g$;

$$E^2 = (pc)^2 + \left(mc^2\right)^2$$

$$g^2\left(mc^2\right)^2 = \frac{\left(mc^2\right)^2}{1 - c^2/v_p^2} = \left(\frac{hc}{l}\right)^2 + \left(mc^2\right)^2.$$

Dividing by $(mc^2)^2$ leads to

$$1 - \frac{c^2}{v_p^2} = \frac{1}{1 + h^2/(mc\,l)^2}, \qquad \text{so that}$$

$$\frac{c^2}{v_p^2} - 1 = \frac{1}{1 + h^2/(mc\,l)^2} = \frac{h^2(mc\,l)^2}{h^2(mc\,l)^2 + 1} = \frac{1}{1 + (mc\,l)^2/h^2},$$

which is an equivalent statement of the desired result.

It should be noted that in the first method presented above could be used to find l in terms of v_p directly, and in the second method the energy could be found in terms of v_g. The final result is, or course, the same.

(b) Using the result of part (a),

$$v_p = c\sqrt{1+\left(\frac{(9.1\times10^{-31}\ \text{kg})(3.0\times10^{8}\ \text{m/s})(1.0\times10^{-13}\ \text{m})}{6.63\times10^{-34}\ \text{J}\cdot\text{s}}\right)^2} = 1.00085c,$$

and $v_g = c^2/v_p = 0.99915c$.

For a calculational shortcut, write the result of part (a) as

$$v_p = c\sqrt{1+\left(\frac{mc^2 l}{hc}\right)^2} = c\sqrt{1+\left(\frac{(511\times10^{3}\ \text{eV})(1.00\times10^{-13}\ \text{m})}{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}\right)^2} = 1.00085c.$$

In both of the above answers, the statement that the de Broglie wavelength is "exactly" 10^{-13} m means that the answers can be given to any desired precision.

23. What effect on the scattering angle in the Davisson-Germer experiment does increasing the electron energy have?

【Sol】

Increasing the electron energy increases the electron's momentum, and hence decreases the electron's de Broglie wavelength. From Equation (2.13), a smaller de Broglie wavelength results in a smaller scattering angle.

25. In Sec. 3.5 it was mentioned that the energy of an electron entering a crystal increase, which reduces its de Broglie wavelength. Consider a beam of 54-eV electrons directed at a nickel target. The potential energy of an electron that enters the target changes by 26 eV. (a) Compare the electron speeds outside and inside the target. (b) Compare the respective de Broglie wavelengths.

【Sol】

(a) For the given energies, a nonrelativistic calculation is sufficient;

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(54\text{ eV})(1.60\times10^{-19}\text{ J/eV})}{9.1\times10^{-31}\text{ kg}}} = 4.36\text{ m/s}$$

outside the crystal, and (from a similar calculation, with K = 80 eV), v = 5.30 x 10^6 m/s inside the crystal (keeping an extra significant figure in both calculations).

(b) With the speeds found in part (a), the de Brogile wavelengths are found from

$$l = \frac{h}{p} = \frac{h}{mv} = \frac{6.63\times10^{-34}\text{ J}\cdot\text{s}}{(9.11\times10^{-31}\text{ kg})(4.36\times10^{6}\text{ m/s})} = 1.67\times10^{-10}\text{ m},$$

or 0.167 nm outside the crystal, with a similar calculation giving 0.137 nm inside the crystal.

27. Obtain an expression for the energy levels (in MeV) of a neutron confined to a one-dimensional box 1.00 x 10^{-14} m wide. What is the neutron's minimum energy? (The diameter of an atomic nucleus is of this order of magnitude.)

【Sol】

From Equation (3.18),

$$E_n = n^2 \frac{h^2}{8mL^2} = n^2 \frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(1.67\times10^{-27}\ \text{kg})(1.00\times10^{-14}\ \text{m})^2} = n^2 3.28\times10^{-13}\ \text{J} = n^2 20.5\ \text{MeV}.$$

The minimum energy, corresponding to n = 1, is 20.5 MeV

29. A proton in a one-dimensional box has an energy of 400 keV in its first excited state. How wide is the box?

【Sol】

The first excited state corresponds to n = 2 in Equation (3.18). Solving for the width L,

$$L = n\sqrt{\frac{h^2}{8mE_2}} = 2\sqrt{\frac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(1.67\times10^{-27}\ \text{kg})(400\times10^3\ \text{eV})(1.60\times10^{-19}\ \text{J/eV})}}$$

$$= 4.53\times10^{-14}\ \text{m} = 45.3\ \text{fm}.$$

31. The atoms in a solid possess a certain minimum **zero-point energy** even at 0 K, while no such restriction holds for the molecules in an ideal gas. Use the uncertainty principle to explain these statements.

【Sol】

Each atom in a solid is limited to a certain definite region of space - otherwise the assembly of atoms would not be a solid. The uncertainty in position of each atom is therefore finite, and its momentum and hence energy cannot be zero. The position of an ideal-gas molecule is not restricted, so the uncertainty in its position is effectively infinite and its momentum and hence energy can be zero.

33. The position and momentum of a 1.00-keV electron are simultaneously determined. If its position is located to within 0.100 nm, what is the percentage of uncertainty in its momentum?

【Sol】

The percentage uncertainty in the electron's momentum will be at least

$$\frac{\Delta p}{p} = \frac{h}{4pp\Delta x} = \frac{h}{4p\Delta x\sqrt{2mK}} = \frac{hc}{4p\Delta x\sqrt{2(mc)^2 K}}$$

$$= \frac{(1.24\times10^{-6}\ \text{eV}\cdot\text{m})}{4p(1.00\times10^{-10}\ \text{m})\sqrt{2(511\times10^{3}\ \text{eV})(1.00\times10^{3}\ \text{eV})}} = 3.1\times10^{-2} = 3.1\,\%.$$

Note that in the above calculation, conversion of the mass of the electron into its energy equivalent in electronvolts is purely optional; converting the kinetic energy into joules and using h = 6.626 x 10^{-34} J· s will of course give the same percentage uncertainty.

35. How accurately can the position of a proton with $v << c$ be determined without giving it more than 1.00 keV of kinetic energy?

【Sol】

The proton will need to move a minimum distance

$$v\Delta t \geq v\frac{h}{4p\Delta E},$$

where v can be taken to be

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\Delta E}{m}}, \quad \text{so that}$$

$$v\Delta t = \sqrt{\frac{2K}{m}}\frac{h}{4p\Delta E} = \frac{h}{2p\sqrt{2mK}} = \frac{hc}{2p\sqrt{2(mc^2)K}}$$

$$= \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{2p\sqrt{2(938\times10^6\ \text{MeV})(1.00\times10^3\ \text{eV})}} = 1.44\times10^{-13}\ \text{m} = 0.144\ \text{pm}.$$

(See note to the solution to Problem 3-33 above).

The result for the product $v\Delta t$ may be recognized as $v\Delta t \geq h/2pp$; this is not inconsistent with Equation (3.21), $\Delta x\,\Delta p \geq h/4p$. In the current problem, ΔE was taken to be the (maximum) kinetic energy of the proton. In such a situation,

$$\Delta E = \frac{\Delta\left(p^2\right)}{m} = 2\frac{p}{m}\Delta p = 2v\Delta p,$$

which is consistent with the previous result.

37. A marine radar operating at a frequency of 9400 MHz emits groups of electromagnetic waves 0.0800 μs in duration. The time needed for the reflections of these groups to return indicates the distance to a target. (a) Find the length of each group and the number of waves it contains. (b) What is the approximate minimum bandwidth (that is, spread of frequencies) the radar receiver must be able to process?

【Sol】

(a) The length of each group is

$$c\Delta t = (3.0\times10^{8}\ \text{m/s})(8.0\times10^{-5}\ \text{s}) = 24\ \text{m}.$$

The number of waves in each group is the pulse duration divided by the wave period, which is the pulse duration multiplied by the frequency,

$$(8.0\times10^{-8}\ \text{s})(4900\times10^{6}\ \text{Hz}) = 752\ \text{waves}.$$

(b) The bandwidth is the reciprocal of the pulse duration,

$$\left(8.0\times10^{-8}\ \text{s}\right)^{-1} = 12.5\ \text{MHz}.$$

39. The frequency of oscillation of a harmonic oscillator of mass m and spring constant C is $n = \sqrt{C/m}\,/2p$. The energy of the oscillator is $E = p^2/2m + Cx^2/2$, where p is its momentum when its displacement from the equilibrium position is x. In classical physics the minimum energy of the oscillator is $E_{\min} = 0$. Use the uncertainty principle to find an expression for E in terms of x only and show that the minimum energy is actually $E_{\min} = hn/2$ by setting $dE/dx = 0$ and solving for $E_{\min}$.

【Sol】

To use the uncertainty principle, make the identification of p with Δp and x with Δx, so that $p = h/(4\pi x)$, and

$$E = E(x) = \left(\frac{h^2}{8p^2 m}\right)\frac{1}{x^2} + \left(\frac{C}{2}\right)x^2.$$

Differentiating with respect to x and setting $\frac{d}{dx}E = 0$,

$$-\left(\frac{h^2}{4p^2 m}\right)\frac{1}{x^3} + Cx = 0,$$

which is solved for

$$x^2 = \frac{h}{2p\sqrt{mC}}.$$

Substution of this value into $E(x)$ gives

$$E_{\min} = \left(\frac{h^2}{8p^2 m}\right)\left(\frac{2p\sqrt{mC}}{h}\right) + \left(\frac{C}{2}\right)\left(\frac{h}{2p\sqrt{mC}}\right) = \frac{h}{2p}\sqrt{\frac{C}{m}} = \frac{hn}{2}.$$

1. The great majority of alpha particles pass through gases and thin metal foils with no deflections. To what conclusion about atomic structure does this observation lead?

【Sol】

The fact that most particles pass through undetected means that there is not much to deflect these particles; most of the volume of an atom is empty space, and gases and metals are overall electrically neutral.

3. Determine the distance of closest approach of 1.00-MeV protons incident on gold nuclei.

【Sol】

For a "closest approach", the incident proton must be directed "head-on" to the nucleus, with no angular momentum with respect to the nucleus (an "Impact parameter" of zero; see the Appendix to Chapter 4). In this case, at the point of closest approach the proton will have no kinetic energy, and so the potential energy at closest approach will be the initial kinetic energy, taking the potential energy to be zero in the limit of very large separation. Equating these energies,

$$K_{\text{initial}} = \frac{Ze^2}{4pe_o r_{\min}}, \quad \text{or}$$

$$r_{\min} = \left(\frac{1}{4pe_o}\right)\frac{Ze^2}{K_{\text{initial}}} = (8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\frac{(79)(1.60\times10^{-19}\ \text{C})^2}{1.60\times10^{-13}\ \text{J}} = 1.14\times10^{-13}\ \text{m}.$$

5. What is the shortest wavelength present in the Brackett series of spectral lines?

【Sol】

The wavelengths in the Brackett series are given in Equation (4.9); the shortest wavelength (highest energy) corresponds to the largest value of n. For $n \rightarrow \infty$,

$$l \rightarrow \frac{16}{R} = \frac{16}{1.097 \times 10^{7}\ \mathrm{m}^{-1}} = 1.46 \times 10^{-6}\ \mathrm{m} = 1.46\ m\mathrm{m}$$

7. In the Bohr model, the electron is in constant motion. How can such an electron have a negative amount of energy?

【Sol】

While the kinetic energy of any particle is positive, the potential energy of any pair of particles that are mutually attracted is negative. For the system to be bound, the total energy, the sum of the positive kinetic energy and the total negative potential energy, must be negative. For a classical particle subject to an inverse-square attractive force (such as two oppositely charged particles or two uniform spheres subject to gravitational attraction in a circular orbit, the potential energy is twice the negative of the kinetic energy.

9. The fine **structure constant** is defined as $a = e^2/2e_o hc$. This quantity got its name because it first appeared in a theory by the German physicist Arnold Sommerfeld that tried to explain the fine structure in spectral lines (multiple lines close together instead of single lines) by assuming that elliptical as well as circular orbits are possible in the Bohr model. Sommerfeld's approach was on the wrong track, but a has nevertheless turned out to be a useful quantity in atomic physics. (a) Show that $a = v_1/c$, where v, is the velocity of the electron in the ground state of the Bohr atom. (b) Show that the value of a is very close to 1/137 and is a pure number with no dimensions. Because the magnetic behavior of a moving charge depends on its velocity, the small value of a is representative of the relative magnitudes of the magnetic and electric aspects of electron behavior in an atom. (c) Show that $a a_o = l_c/2p$, where a_o is the radius of the ground-state Bohr orbit and l_c is the Compton wavelength of the electron.

【Sol】

(a) The velocity v, is given by Equation (4.4), with $r = r_1 = a_o$. Combining to find v_1^2,

$$v_1^2 = \frac{e^2}{4pe_o m a_o} = \frac{e^2}{4pe_o m\left(\dfrac{h^2 e_o}{pme^2}\right)} = \frac{e^4}{4e_o^2 h^2}, \quad \text{so} \quad \frac{v_1}{c} = \frac{e^2}{2e_o h}\frac{1}{c} = a\,.$$

(b) From the above,

$$a = \frac{\left(1.60\times10^{-19}\ \mathrm{C}\right)^2}{2\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)\left(6.63\times10^{-34}\ \mathrm{J\cdot s}\right)\left(3.00\times10^{8}\ \mathrm{m/s}\right)} = 7.30\times10^{-3},$$

so that 1/α = 137.1 to four significant figures.

A close cheek of the units is worthwhile; treating the units as algebraic quantities the units as given in the above calculation are

$$\frac{[\mathrm{C}^2]}{\dfrac{[\mathrm{C}^2]}{[\mathrm{N}][\mathrm{m}^2]}[\mathrm{J}][\mathrm{s}]\dfrac{[\mathrm{m}]}{[\mathrm{s}]}} = \frac{[\mathrm{N}\cdot\mathrm{m}]}{[\mathrm{J}]} = 1.$$

Thus, α is a dimensionless quantity, and will have the same numerical value in any system of units. The most accurate (November, 2001) value of 1/α is

$$\frac{1}{a} = 137.03599976,$$

accurate to better than 4 parts per billion.

(c) Using the above expression for α and Equation (4.13) with n = 1 for a_o,

$$a a_o = \frac{e^2}{2e_o hc}\frac{h^2 e_o}{p m e^2} = \frac{1}{2p}\frac{h}{mc} = \frac{l_C}{2p},$$

where the Compton wavelength l_C is given by Equation (2.22).

11. Find the quantum number that characterizes the earth's orbit around the sun. The earth's mass is 6.0 x 10^{24} kg, its orbital radius is 1.5 x 10^{11} m, and its orbital speed is 3.0 **x** 10^{4} m/s.

【Sol】

With the mass, orbital speed and orbital radius of the earth known, the earth's orbital angular momentum is known, and the quantum number that would characterize the earth's orbit about the sun would be this angular momentum divided by ;

$$n = \frac{L}{\hbar} = \frac{mvR}{\hbar} = \frac{(6.0\times10^{24}\ \text{kg})(3.0\times10^{4}\ \text{m/s})(1.5\times10^{11}\ \text{m})}{1.06\times10^{-34}\ \text{J}\cdot\text{s}} = 2.6\times10^{74}.$$

(The number of significant figures not of concern.)

13. Compare the uncertainty in the momentum of an electron confined to a region of linear dimension a_o with the momentum of an electron in a ground-state Bohr orbit.

【Sol】

The uncertainty in position of an electron confined to such a region is, from Equation (3.22), $\Delta p \geq \ /2a_o$, while the magnitude of the linear momentum of an electron in the first Bohr orbit is

$$p = \frac{h}{l} = \frac{h}{2pa_o} = \frac{\hbar}{a_o};$$

the value of Δp found from Equation (3.13) is half of this momentum.

15. What effect would you expect the rapid random motion of the atoms of an excited gas to have on the spectral lines they produce?

【Sol】

The Doppler effect shifts the frequencies of the emitted light to both higher and lower frequencies to produce wider lines than atoms at rest would give rise to.

17. A proton and an electron, both at rest initially, combine to form a hydrogen atom in the ground state. A single photon is emitted in this process. What is its wavelength?

【Sol】

It must assumed that the initial electrostatic potential energy is negligible, so that the final energy of the hydrogen atom is E_1 = -13.6 eV. The energy of the photon emitted is then -E_1, and the wavelength is

$$l = \frac{hc}{-E_1} = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{13.6\ \text{eV}} = 9.12\times10^{-8}\ \text{m} = 91.2\ \text{nm},$$

in the ultraviolet part of the spectrum (see, for instance, the back endpapers of the text).

19. Find the wavelength of the spectral line that corresponds to a transition in hydrogen from the n = 10 state to the ground state. In what part of the spectrum is this?

【Sol】

From either Equation (4.7) with n = 10 or Equation (4.18) with n_f = 1 and n_i = 10,

$$l = \frac{100}{99}\frac{1}{R} = \frac{100}{99}\frac{1}{1.097\times10^{7}\ \text{m}^{-1}} = 9.21\times10^{-8}\ \text{m} = 92.1\ \text{nm},$$

which is in the ultraviolet part of the spectrum (see, for instance, the back endpapers of the text).

21. A beam of electrons bombards a sample of hydrogen. Through what potential difference must the electrons have been accelerated if the first line of the Balmer series is to be emitted?

【Sol】

The electrons' energy must be at least the difference between the n = 1 and n = 3 levels,

$$\Delta E = E_3 - E_1 = -E_1\left(1-\frac{1}{9}\right) = (13.6\ \text{eV})\,\frac{8}{9} = 12.1\ \text{eV}$$

(this assumes that few or none of the hydrogen atoms had electrons in the n = 2 level). A potential difference of 12.1 eV is necessary to accelerate the electrons to this energy.

23. The longest wavelength in the Lyman series is 121.5 nm and the shortest wavelength in the Balmer series is 364.6 nm. Use the figures to find the longest wavelength of light that could ionize hydrogen.

【Sol】

The energy needed to ionize hydrogen will be the energy needed to raise the energy from the ground state to the first excited state plus the energy needed to ionize an atom in the second excited state; these are the energies that correspond to the longest wavelength (least energetic photon) in the Lyman series and the shortest wavelength (most energetic photon) in the Balmer series. The energies are proportional to the reciprocals of the wavelengths, and so the wavelength of the photon needed to ionize hydrogen is

$$l = \left(\frac{1}{l_{2\to 1}} + \frac{1}{l_{\infty \to 2}} \right)^{-1} = \left(\frac{1}{121.5\,\text{nm}} + \frac{1}{364.6\,\text{nm}} \right)^{-1} = 91.13\,\text{nm}.$$

As a check, note that this wavelength is R^{-1}.

25. An excited hydrogen atom emits a photon of wavelength l in returning to the ground state. (a) Derive a formula that gives the quantum number of the initial excited state in terms of l and R. (b) Use this formula to find n_i for a 102.55-nm photon.

【Sol】

(a) From Equation (4.7) with $n = n_i$, $\frac{1}{l} = R\left(1 - \frac{1}{n_i^2}\right)$, which is solved for

$$n_i = \left(1 - \frac{1}{lR}\right)^{-1/2} = \sqrt{\frac{lR}{lR - 1}}.$$

(b) Either of the above forms gives n very close (four place) to 3; specifically, with the product $lR = (102.55\text{x}10^{-9}\text{ m})(1.097\text{x}10^{7}\text{ m}^{-1}) = 1.125$ rounded to four places as 9/8, $n = 3$ exactly.

27. When an excited atom emits a photon, the linear momentum of the photon must be balanced by the recoil momentum of the atom. As a result, some of the excitation energy of the atom goes into the kinetic energy of its recoil. (a) Modify Eq. (4.16) to include this effect. (b) Find the ratio between the recoil energy and the photon energy for the $n = 3 \rightarrow n = 2$ transition in hydrogen, for which E_f - E_i = 1.9 eV. Is the effect a major one? A nonrelativistic calculation is sufficient here.

【Sol】

(a) A relativistic calculation would necessarily involve the change in mass of the atom due to the change in energy of the system. The fact that this mass change is too small to measure (that is, the change is measured indirectly by measuring the energies of the emitted photons) means that a nonrelativistic calculation should suffice. In this situation, the kinetic energy of the recoiling atom is

$$K = \frac{p^2}{2M} = \frac{(hn/c)^2}{2M},$$

where m is the ftequency of the emitted photon and $p = h/l = hn/c$ is the magnitude of the momentum of both the photon and the recoiling atom. Equation (4.16) is then

$$E_i - E_f = h\nu + K = h\nu + \frac{(h\nu)^2}{2Mc^2} = h\nu\left(1 + \frac{h\nu}{2Mc^2}\right).$$

This result is equivalent to that of Problem 2-53, where $h\nu = E_\infty$. and the term $p^2/(2M)$ corresponds to $E_\infty - E$ in that problem. As in Problem 2-53, a relativistic calculation is manageable; the result would be

$$E_f - E_i = h\nu\left(1 + \frac{1}{2}\left(1 + \frac{Mc^2}{h\nu}\right)^{-1}\right),$$

a form not often useful; see part (b).

(b) As indicated above and in the problem statement, a nonrelativistle calculation is sufficient. As in part (a),

$$K = \frac{p^2}{2M} = \frac{(\Delta E / c)^2}{2M}, \quad \text{and} \quad \frac{K}{\Delta E} = \frac{\Delta E}{2Mc^2} = \frac{1.9\ \text{eV}}{2(939 \times 10^6\ \text{eV})} = 1.01 \times 10^{-9},$$

or 1.0 x 10^{-9} to two significant figures. In the above, the rest energy of the hydrogen atom is from the front endpapers.

29. Show that the frequency of the photon emitted by a hydrogen atom in going from the level $n + 1$ to the level n is always intermediate between the frequencies of revolution of the electron in the respective orbits.

【Sol】

There are many equivalent algebraic methods that may be used to derive Equation (4.19), and that result will be cited here;

$$f_n = -\frac{2E_1}{h}\frac{1}{n^3}.$$

The frequency ν of the photon emitted in going from the level $n + 1$ to the level n is obtained from Equation (4.17) with $n_i = n + 1$ and $n_f = n$;

$$n = \frac{\Delta E}{h} = \left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] = -\frac{2E_1}{h}\left[\frac{n+\frac{1}{2}}{n^2(n+1)^2}\right].$$

This can be seen to be equivalent to the expression for v in terms of n and p that was found in the derivation of Equation (4.20), but with n replaced by $n + 1$ and $p = 1$. Note that in this form, ν is positive because E_1 is negative. From this expression

$$n = -\frac{2E_1}{hn^3}\left[\frac{n^2+\frac{1}{2}n}{n^2+2n+1}\right] = f_n\left[\frac{n^2+\frac{1}{2}n}{n^2+2n+1}\right] < f_n,$$

as the term in brackets is less than 1. Similarly,

$$n = -\frac{2E_1}{h(n+1)^3}\left[\frac{(n+\frac{1}{2})(n+1)}{n^2}\right] = f_{n+1}\left[\frac{(n+\frac{1}{2})(n+1)}{n^2}\right] > f_{n+1},$$

as the term in brackets is greater than 1.

31. A μ⁻ muon is in the n = 2 state of a muonic atom whose nucleus is a proton. Find the wavelength of the photon emitted when the muonic atom drops to its ground state. In what part of the spectrum is this wavelength?

【Sol】

For a muonic atom, the Rydberg constant is multiplied by the ratio of the reduced masses of the muoninc atom and the hydrogen atom, $R' = R\,(m'/m_e) = 186R$, as in Example 4.7; from Equation (4.7),

$$l = \frac{4/3}{R'} = \frac{4/3}{186(1.097\times 10^{7}\ \text{m}^{-1})} = 6.53\times 10^{-10}\ \text{m} = 0.653\,\text{nm},$$

in the x-ray range.

$$m_m = 207m_e, \quad m_p = 1836m_e \qquad m' = \frac{m_m m_p}{m_m + m_p} = 186m_e$$

33. A mixture of ordinary hydrogen and tritium, a hydrogen isotope whose nucleus is approximately 3 times more massive than ordinary hydrogen, is excited and its spectrum observed. How far apart in wavelength will the H_α lines of the two kinds of hydrogen be?

【Sol】

The H_α lines, corresponding to $n = 3$ in Equation (4.6), have wavelengths of $l = (36/5)\,(1/R)$. For a tritium atom, the wavelength would be $l_T = (36/5)\,(1/RT)$, where RT is the Rydberg constant evaluated with the reduced mass of the tritium atom replacing the reduced mass of the hydrogen atom. The difference between the wavelengths would then be

$$\Delta l = l - l_T = l\left[1 - \frac{l_T}{l}\right] = l\left[1 - \frac{R}{R_T}\right].$$

The values of R and RT are proportional to the respective reduced masses, and their ratio is

$$\frac{R}{R_T} = \frac{m_e m_H \,/\,(m_e + m_H)}{m_e m_T \,/\,(m_e + m_T)} = \frac{m_H(m_e + m_T)}{m_T(m_e + m_H)}.$$

Using this in the above expression for $\Delta\lambda$,

$$\Delta l = l\left[\frac{m_e(m_T - m_H)}{m_e(m_e + m_H)}\right] \approx l\,\frac{2m_e}{3m_H},$$

where the approximations $m_e + rn_H$ » m_H and m_T » $3m_H$ have been used. Inserting numerical values,

$$\Delta l = \frac{(36/5)}{(1.097\times10^{7}\ \text{m}^{-1})}\,\frac{2(9.11\times10^{-31}\ \text{kg})}{3(1.67\times10^{-27}\ \text{kg})} = 2.38\times10^{-10}\ \text{m} = 0.238\,\text{nm}.$$

35. (a) Derive a formula for the energy levels of a **hydrogenic atom,** which is an ion such as He^+ or Li^{2+} whose nuclear charge is $+Ze$ and which contains a single electron. (b) Sketch the energy levels of the He' ion and compare them with the energy levels of the H atom. (c) An electron joins a bare helium nucleus to form a He^+ ion. Find the wavelength of the photon emitted in this process if the electron is assumed to have had no kinetic energy when it combined with the nucleus.

【Sol】

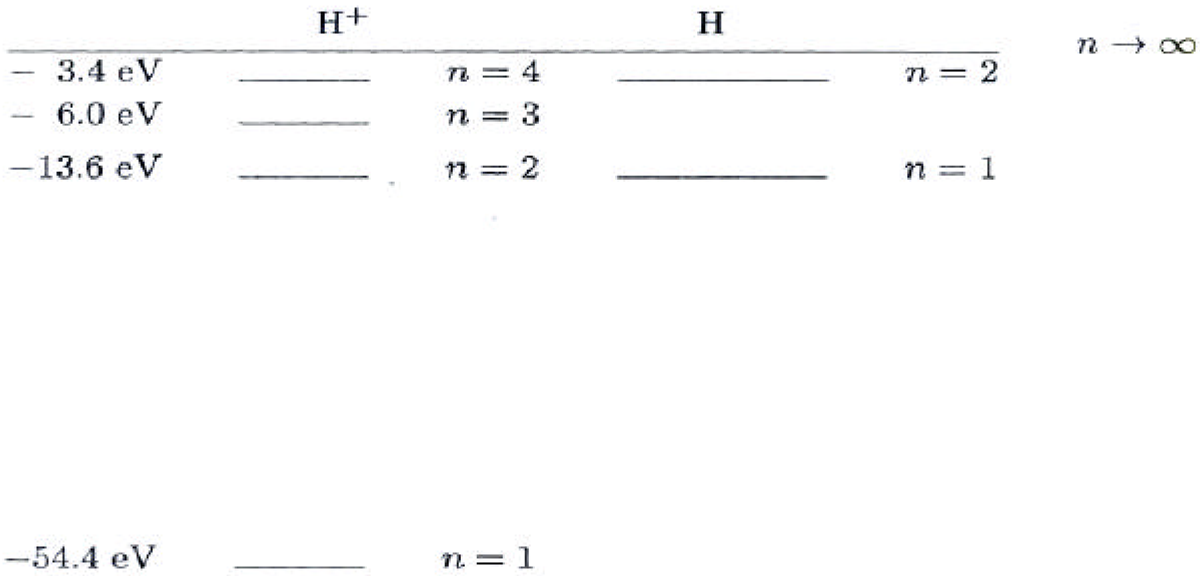
(a) The steps leading to Equation (4.15) are repeated, with Ze^2 instead of e^2 and Z^2e^4 instead of e^4, giving

$$E_n = -\frac{m' Z^2 e^4}{8pe_o^2 h^2}\frac{1}{n^2},$$

where the reduced mass m' will depend on the mass of the nucleus.

(b) A plot of the energy levels is given below. The scale is close, but not exact, and of course there are many more levels corresponding to higher n. In the approximation that the reduced masses are the same, for He^+, with $Z = 2$, the $n = 2$ level is the same as the $n = 1$ level for Hydrogen, and the $n = 4$ level is the same as the $n = 2$ level for hydrogen.

The energy levels for H and He^+:



(c) When the electron joins the Helium nucleus, the electron-nucleus system loses energy; the emitted photon will have lost energy ΔE = 4 (-13.6 eV) = -54.4 eV, where the result of part (a) has been used. The emitted photon's wavelength is

$$l = \frac{hc}{-\Delta E} = \frac{1.24\times 10^{-6}\ \text{eV}\cdot\text{m}}{54.4\,\text{eV}} = 2.28\times 10^{-8}\ \text{m} = 22.8\ \text{nm}.$$

37. A certain ruby laser emits 1.00-J pulses of light whose wavelength is 694 nm. What is the minimum number of Cr^{3+} ions in the ruby?

【Sol】

The minimum number of Cr^{3+} ions will he the minimum number of photons, which is the total energy of the pulse divided by the energy of each photon,

$$\frac{E}{hc/l} = \frac{El}{hc} = \frac{(1.00\text{ J})(694\times10^{-9}\text{ m})}{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.0\times10^{8}\text{ m/s})} = 3.49\times10^{18}\text{ ions.}$$

39. The Rutherford scattering formula fails to agree with the data at very small scattering angles. Can you think of a reason?

【Sol】

Small angles correspond to particles that are not scattered much at all, and the structure of the atom does not affect these particles. To these nonpenetrating particles, the nucleus is either partially or completely screened by the atom's electron cloud, and the scattering analysis, based on a pointlike positively charged nucleus, is not applicable.

41. A 5.0-MeV alpha particle approaches a gold nucleus with an impact parameter of 2.6 x 10^{-13} m. Through what angle will it be scattered?

【Sol】

From Equation (4.29), using the value for $1/4pe_{o}$ given in the front endpapers,

$$\cot\frac{q}{2} = \frac{(5.0\,\text{eV})(1.60\times10^{-13}\ \text{J/MeV})}{(8.99\times10^{9}\ \text{N}\cdot\text{m}^{2}/\text{C}^{2})(79)(1.60\times10^{-19}\ \text{C})^{2}}(2.6\times10^{-13}\ \text{m}) = 11.43,$$

keeping extra significant figures. The scattering angle is then

$$q = 2\cot^{-1}(11.43) = 2\tan^{-1}\left(\frac{1}{11.43}\right) = 10^{o}.$$

43. What fraction of a beam of 7.7-MeV alpha particles incident upon a gold foil 3.0 x 10^{-7} m thick is scattered by less than 1^{o}?

【Sol】

The fraction scattered by less than 1^{o} is 1 - *f*, with *f* given in Equation (4.31);

$$f = pnt\left(\frac{Ze^2}{4pe_o K}\right)^2 \cot^2\frac{q}{2} = pnt\left(\frac{1}{4pe_o}\right)^2\left(\frac{Ze^2}{K}\right)^2 \cot^2\frac{q}{2}$$

$$= p(5.90\times10^{28}\ \text{m}^{-3})(3.0\times10^{-7}\ \text{m})(9.0\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2)^2$$

$$\times\left(\frac{(79)(1.6\times10^{-19}\text{C})^2}{(7.7\text{MeV})(1.6\times10^{-13}\text{J/MeV})}\right)^2 \cot^2(0.5^o) = 0.16,$$

where n, the number of gold atoms per unit volume, is from Example 4.8. The fraction scattered by less than 1^o is $1 - f = 0.84$.

45. Show that twice as many alpha particles are scattered by a foil through angles between 60^o and 90^o as are scattered through angles of 90^o or more.

【Sol】

Regarding f as a function of 0 in Equation (4.31), the number of particles scattered between 60^o and 90^o is $f(60^o) - f(90^o)$, and the number scattered through angles greater than 90^o is just $f(90^o)$, and

$$\frac{f(60^o) - f(90^o)}{f(90^o)} = \frac{\cot^2(30^o) - \cot^2(45^o)}{\cot^2(45^o)} = \frac{3-1}{1} = 2,$$

so twice as many particles are scattered between 60^o and 90^o than are scattered through angles greater than 90^o.

47. In special relativity, a photon can be thought of as having a "mass" of $m = E_\nu/c^2$. This suggests that we can treat a photon that passes near the sun in the same way as Rutherford treated an alpha particle that passes near a nucleus, with an attractive gravitational force replacing the repulsive electrical force. Adapt Eq. (4.29) to this situation and find the angle of deflection q for a photon that passes $b = R_{sun}$ from the center of the sun. The mass and radius of the sun are respectively 2.0 x 10^{30} kg and 7.0 x 10^8 m. In fact, general relativity shows that this result is exactly half the actual deflection, a conclusion supported by observations made during solar clipses as mentioned in Sec. 1.10.

【Sol】

If gravity acted on photons as if they were massive objects with mass $m = E_\nu/c^2$, the magnitude of the force F in Equation (4.28) would be

$$F = \frac{GM_{sun}m}{r^2};$$

the factors of r^2 would cancel, as they do for the Coulomb force, and the result is

$$2mc^2b\sin\frac{q}{2} = 2GM_{sun}m\cos\frac{q}{2} \quad \text{and} \quad \cot\frac{q}{2} = \frac{c^2b}{GM_{sun}},$$

a result that is independent of the photon's energy. Using $b = R_{sun}$,

$$q = 2\tan^{-1}\left(\frac{GM_{sun}}{c^2R_{sun}}\right) = 2\tan^{-1}\left(\frac{(6.67\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2)(2.0\times10^{30}\,\text{kg})}{(3.0\times10^8\,\text{m/s})(7.0\times10^8\,\text{m})}\right)$$

$$= 2.43\times10^{-4}\ \text{deg} = 0.87''.$$

1. Which of the wave functions in Fig. 5.15 cannot have physical significance in the interval shown? Why not?

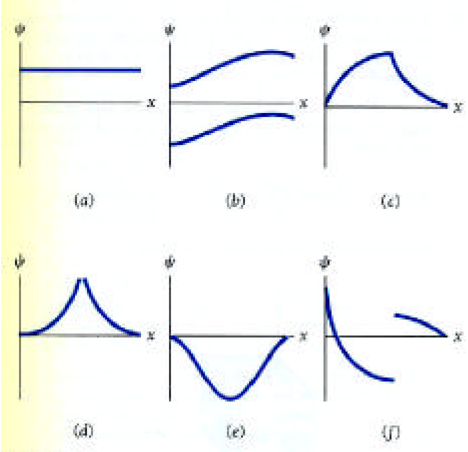


Figure 5.15

【Sol】

Figure (b) is double valued, and is not a function at all, and cannot have physical significance. Figure (c) has discontinuous derivative in the shown interval. Figure (d) is finite everywhere in the shown interval. Figure (f) is discontinuous in the shown interval.

3. Which of the following wave functions cannot be solutions of Schrödinger's equation for all values of x? Why not? (a) $y = A \sec x$; (b) $y = A \tan x$; (c) $y = A \exp(x^2)$; (d) $y = A \exp(-x^2)$.

【Sol】

The functions (a) and (b) are both infinite when $\cos x = 0$, at $x = \pm\pi/2, \pm 3\pi/2, \ldots \pm(2n+1)\pi/2$ for any integer n, neither $y = A \sec x$ or $y = A \tan x$ could be a solution of Schrödinger's equation for all values of x. The function (c) diverges as $x \to \pm\infty$, and cannot be a solution of Schrödinger's equation for all values of x.

5. The wave function of a certain particle is $\psi = A\cos^2 x$ for $-\pi/2 < x < \pi/2$. (a) Find the value of A. (b) Find the probability that the particle be found between $x = 0$ and $x = \pi/4$.

【Sol】

Both parts involve the integral $\int \cos^4 x\,dx$, evaluated between different limits for the two parts. Of the many ways to find this integral, including consulting tables and using symbolic-manipulation programs, a direct algebraic reduction gives

$$\cos^4 x = (\cos^2 x)^2 = \left[\tfrac{1}{2}(1+\cos 2x)\right]^2 = \tfrac{1}{4}\left[1 + 2\cos 2x + \cos^2(2x)\right]$$

$$= \tfrac{1}{4}\left[1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\right] = \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x,$$

where the identity $\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$ has been used twice.

(a) The needed normalization condition is

$$\int_{-\pi/2}^{+\pi/2} \psi^* \psi\,dx = A^2 \int_{-\pi/2}^{+\pi/2} \cos^4 x\,dx$$

$$= A^2\left[\tfrac{3}{8}\int_{-\pi/2}^{+\pi/2} dx + \tfrac{1}{2}\int_{-\pi/2}^{+\pi/2} \cos 2x\,dx + \tfrac{1}{8}\int_{-\pi/2}^{+\pi/2} \cos 4x\,dx\right] = 1$$

The integrals

$$\int_{-\pi/2}^{+\pi/2} \cos 2x\,dx = \tfrac{1}{2}\sin 2x\Big|_{-\pi/2}^{+\pi/2} \quad \text{and} \quad \int_{-\pi/2}^{+\pi/2} \cos 4x\,dx = \tfrac{1}{4}\sin 4x\Big|_{-\pi/2}^{+\pi/2}$$

are seen to be vanish, and the normalization condition reduces to

$$1 = A^2\left(\frac{3}{8}\right)\pi, \quad \text{or} \quad A = \sqrt{\frac{8}{3\pi}}.$$

(b) Evaluating the same integral between the different limits,

$$\int_0^{\pi/4} \cos^4 x\,dx = \left[\tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x\right]_0^{\pi/4} = \frac{3\pi}{32} + \frac{1}{4},$$

The probability of the particle being found between $x = 0$ and $x = \pi/4$ is the product of this integral and A^2, or

$$A^2\left(\frac{3\pi}{32} + \frac{1}{4}\right) = \frac{8}{3\pi}\left(\frac{3\pi}{32} + \frac{1}{4}\right) = 0.46$$

7. As mentioned in Sec. 5.1, in order to give physically meaningful results in calculations a wave function and its partial derivatives must be finite, continuous, and single-valued, and in addition must be normalizable. Equation (5.9) gives the wave function of a particle moving freely (that is, with no forces acting on it) in the $+x$ direction as

$$\Psi = Ae^{-(i/\hbar)(Et - pc)}$$

where E is the particle's total energy and p is its momentum. Does this wave function meet all the above requirements? If not, could a linear superposition of such wave functions meet these requirements? What is the significance of such a superposition of wave functions?

【Sol】

The given wave function satisfies the continuity condition, and is differentiable to all orders with respect to both t and x, but is not normalizable; specifically, $\Psi^*\Psi = A^*A$ is constant in both space and time, and if the particle is to move freely, there can be no limit to its range, and so the integral of $\Psi^*\Psi$ over an infinite region cannot be finite if $A \neq 0$.

A linear superposition of such waves could give a normalizable wave function, corresponding to a real particle. Such a superposition would necessarily have a non- zero Δp, and hence a finite Δx; at the expense of normalizing the wave function, the wave function is composed of different momentum states, and is localized.

9. Show that the expectation values $<px>$ and $<xp>$) are related by

$$<px> - <xp> = \ /i$$

This result is described by saying that p and x do not **commute,** and it is intimately related to the uncertainty principle.

【Sol】

It's crucial to realize that the expectation value $<px>$ is found from the combined operator $\hat{p}\hat{x}$, which, when operating on the wave function $\Psi(x, t)$, corresponds to "multiply by x, differentiate with respect to x and multiply by $/i$," whereas the operator $\hat{x}\hat{p}$ corresponds to "differentiate with respect to x, multiply by $/i$ and multiply by x." Using these operators,

$$(\hat{p}\hat{x})\Psi = \hat{p}(\hat{x}\Psi) = \frac{\hbar}{i}\frac{\partial}{\partial x}(x\Psi) = \frac{\hbar}{i}\left[\Psi + x\frac{\partial}{\partial x}\Psi\right],$$

where the product rule for partial differentiation has been used. Also,

$$(\hat{x}\hat{p})\Psi = \hat{x}(\hat{p}\Psi) = x\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi\right) = \frac{\hbar}{i}\left[x\frac{\partial}{\partial x}\Psi\right].$$

Thus $$(\hat{p}\hat{x} - \hat{x}\hat{p})\Psi = \frac{\hbar}{i}\Psi$$

and $$< px - xp >= \int_{-\infty}^{\infty} \Psi^* \frac{\hbar}{i} \Psi dx = \frac{\hbar}{i} \int_{-\infty}^{\infty} \Psi^* \Psi dx = \frac{\hbar}{i}$$

for $\Psi(x, t)$ normalized.

11. Obtain Schrödinger' s steady-state equation from Eq.(3.5) with the help of de Broglie' s relationship $l = h / mv$ by letting $y = y$ and finding $¶^2 y / ¶x^2$.

【Sol】

Using $\lambda\nu = v_p$ in Equation (3.5), and using y instead of y,

$$y = A\cos\left(2p\left(t - \frac{x}{v_p}\right)\right) = A\cos\left(2pnt - 2p\frac{x}{l}\right).$$

Differentiating twice with respect to x using the chain rule for partial differentiation (similar to Example 5.1),

$$\frac{\partial y}{\partial x} = -A\sin\left(2pnt - 2p\frac{x}{l}\right)\left(-\frac{2p}{l}\right) = \frac{2p}{l}A\sin\left(2pnt - 2p\frac{x}{l}\right),$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{2p}{l}A\cos\left(2pnt - 2p\frac{x}{l}\right)\left(-\frac{2p}{l}\right) = \left(\frac{2p}{l}\right)^2 A\cos\left(2pnt - 2p\frac{x}{l}\right) = -\left(\frac{2p}{l}\right)^2 y$$

The kinetic energy of a nonrelativistic particle is

$$KE = E - U = \frac{p^2}{2m} = \left(\frac{h}{l}\right)^2 \frac{1}{2m}, \qquad \text{so that} \qquad \frac{1}{l^2} = \frac{2m}{h^2}(E - U)$$

Substituting the above expression relating $\frac{\partial^2 y}{\partial x^2}$ and $\frac{1}{l^2}y$

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2p}{l}\right)^2 y = -\frac{8p^2 m}{h^2}(E - U)y = -\frac{2m}{\hbar^2}(E - U)y, \qquad \text{which is Equation (5.32)}$$

13. One of the possible wave functions of a particle in the potential well of Fig. 5.17 is sketched there. Explain why the wavelength and amplitude of &P vary as they do.

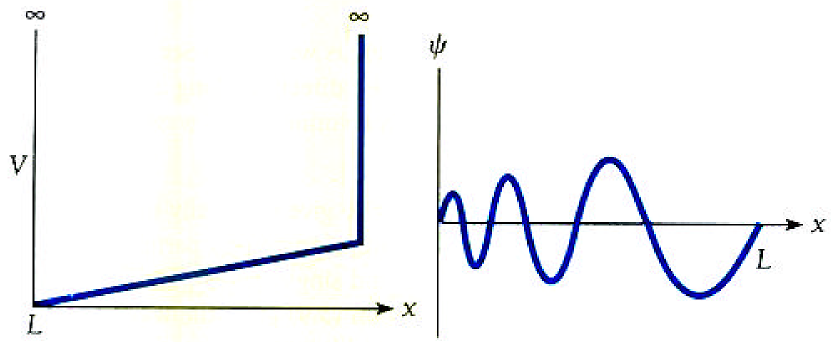


Figure 5.17

【Sol】

The wave function must vanish at $x = 0$, where $V \to \infty$. As the potential energy increases with x, the particle's kinetic energy must decrease, and so the wavelength increases. The amplitude increases as the wavelength increases because a larger wavelength means a smaller momentum (indicated as well by the lower kinetic energy), and the particle is more likely to be found where the momentum has a lower magnitude. The wave function vanishes again where the potential $V \to \infty$; this condition would determine the allowed energies.

15. An important property of the eigenfunctions of a system is that they are **orthogonal** to one another, which means that

$$\int_{-\infty}^{+\infty} \psi_n \psi_m \, dV = 0 \qquad n \neq m$$

Verify this relationship for the eigenfunctions of a particle in a one-dimensional box given by Eq. (5.46).

【Sol】

The necessary integrals are of the form

$$\int_{-\infty}^{+\infty} \psi_n \psi_m \, dx = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$

for integers n, m, with $n \neq m$ and $n \neq -m$. (A more general orthogonality relation would involve the integral of $\psi_n^* \psi_m$, but as the eigenfunctions in this problem are real, the distinction need not be made.)

To do the integrals directly, a convenient identity to use is

$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

as may be verified by expanding the cosines of the sum and difference of α and β. To show orthogonality, the stipulation $n \neq m$ means that $\alpha \neq \beta$ and $\alpha \neq -\beta$ and the integrals are of the form

$$\int_{-\infty}^{+\infty} y_n y_m dx = \frac{1}{L}\int_0^L \left[\cos\frac{(n-m)px}{L} - \cos\frac{(n+m)px}{L}\right] dx$$

$$= \left[\frac{L}{(n-m)p}\sin\frac{(n-m)px}{L} - \frac{L}{(n+m)p}\sin\frac{(n+m)px}{L}\right]_o^L = 0,$$

where $\sin(n - m)\pi = \sin(n - m)\pi = \sin 0 = 0$ has been used.

17. As shown in the text, the expectation value $<x>$ of a particle trapped in a box L wide is $L/2$, which means that its average position is the middle of the box. Find the expectation value $<x^2>$.

【Sol】

Using Equation (5.46), the expectation value $<x^2>$ is

$$< x^2 >_n = \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{npx}{L}\right) dx .$$

See the end of this chapter for an alternate analytic technique for evaluating this integral using *Leibniz's Rule*. From either a table or repeated integration by parts, the indefinite integral is

$$\int x^2 \sin^2 \frac{npx}{L} dx = \left(\frac{L}{np}\right)^3 \int u^3 \sin u du = \left(\frac{L}{np}\right)^3 \left[\frac{u^3}{6} - \frac{u^2}{4}\sin 2u - \frac{u}{4}\cos 2u + \frac{1}{8}\sin 2u\right].$$

where the substitution $u = (n\pi/L)x$ has been made.

This form makes evaluation of the definite integral a bit simpler; when $x = 0$ $u = 0$, and when $x = L$ $u = n\pi$. Each of the terms in the integral vanish at $u = 0$, and the terms with sin $2u$ vanish at $u = n\pi$, $\cos 2u = \cos 2n\pi = 1$, and so the result is

$$< x^2 >_n = \frac{2}{L}\left(\frac{L}{np}\right)^3\left[\frac{(np)^3}{6} - \frac{np}{4}\right] = L^2\left[\frac{1}{3} - \frac{1}{2n^2p^2}\right].$$

As a check, note that

$$\lim_{n \to \infty} < x^2 >_n = \frac{L^2}{3},$$

which is the expectation value of $<x^2>$ in the classical limit, for which the probability distribution is independent of position in the box.

19. Find the probability that a particle in a box L wide can be found between $x = 0$ and $x = L/n$ when it is in the nth state.

【Sol】

This is a special case of the probability that such a particle is between x_1 and x_2, as found in Example 5.4. With $x_1 = 0$ and $x_2 = L$,

$$P_{0L} = \left[\frac{x}{L} - \frac{1}{2np}\sin\frac{2npx}{L}\right]_0^L = \frac{1}{n}.$$

21. A particle is in a cubic box with infinitely hard walls whose edges are L long (Fig. 5. 18). The wave functions of the particle are given by

$$y = A \sin \frac{n_x p x}{L} \sin \frac{n_z p y}{L} \sin \frac{n_z p z}{L} \qquad \begin{array}{l} n_x = 1, 2, 3, \ldots \\ n_y = 1, 2, 3, \ldots \\ n_z = 1, 2, 3, \ldots \end{array}$$

Find the value of the normalization constant A.

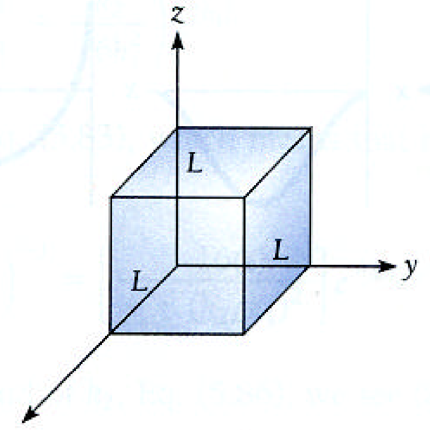


Figure 5.18 A cubic box.

【Sol】

The normalization constant, assuming A to be real, is given by

$$\int y^* y \, dV = 1 = \int y^* y \, dx\, dy\, dz$$

$$= A^2 \left(\int_0^L \sin^2 \frac{n_x p x}{L} dx \right) \left(\int_0^L \sin^2 \frac{n_y p y}{L} dy \right) \left(\int_0^L \sin^2 \frac{n_z p z}{L} dz \right)$$

Each integral above is equal to $L/2$ (from calculations identical to Equation (5.43)). The result is

$$A^2 \left(\frac{L}{2} \right)^3 = 1 \quad \text{or} \quad A = \left(\frac{2}{L} \right)^{3/2}$$

23. (a) Find the possible energies of the particle in the box of Exercise 21 by substituting its wave function y in Schrödinger's equation and solving for E. (Hint: inside the box U = 0.)
(b) Compare the ground-state energy of a particle in a one-dimensional box of length L with that of a particle in the three-dimensional box.

【Sol】

(a) For the wave function of Problem 5-21, Equation (5.33) must be used to find the energy. Before substitution into Equation (5.33), it is convenient and useful to note that for this wave function

$$\frac{\partial^2 y}{\partial x^2} = -\frac{n_x^2 p^2}{L^2} y, \quad \frac{\partial^2 y}{\partial y^2} = -\frac{n_y^2 p^2}{L^2} y, \quad \frac{\partial^2 y}{\partial z^2} = -\frac{n_z^2 p^2}{L^2} y.$$

Then, substitution into Equation (5.33) gives

$$-\frac{p^2}{L^2}(n_x^2 + n_y^2 + n_z^2) y + \frac{2m}{\hbar^2} E y = 0,$$

and so the energies are

$$E_{n_x, n_y, n_z} = \frac{p^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2).$$

(b) The lowest energy occurs when $n_x = n_y = n_z = 1$. None of the integers n_x, n_y, or n_z can be zero, as that would mean $y = 0$ identically. The minimum energy is then

$$E_{min} = \frac{3p^2 \hbar^2}{2mL^2},$$

which is three times the ground-state energy of a particle in a one-dimensional box of length L.

25. A beam of electrons is incident on a barrier 6.00 eV high and 0.200 nm wide. Use Eq. (5.60) to find the energy they should have if 1.00 percent of them are to get through the barrier.

【Sol】

Solving equation (5.60) for k_2,

$$k_2 = \frac{1}{2L}\ln\frac{1}{T} = \frac{1}{2(0.200\times10^{-9}\ \text{m})}\ln(100) = 1.15\times10^{10}\ \text{m}^{-1}$$

Equation (5.86), from the appendix, may be solved for the energy E, but a more direct expression is

$$E = U - KE = U - \frac{p^2}{2m} = U - \frac{(\hbar k_2)^2}{2m}$$

$$= 6.00\ \text{eV} - \frac{\left((1.05\times10^{-34}\,\text{J}\cdot\text{s})(1.15\times10^{10}\,\text{m}^{-1})\right)^2}{2(9.1\times10^{-31}\,\text{kg})(1.6\times10^{-19}\,\text{J/eV})} = 0.95\ \text{eV}$$

27. What bearing would you think the uncertainty principle has on the existence of the zero-point energy of a harmonic oscillator?

【Sol】

If a particle in a harmonic-oscillator potential had zero energy, the particle would have to be at rest at the position of the potential minimum. The uncertainty principle dictates that such a particle would have an infinite uncertainty in momentum, and hence an infinite uncertainty in energy. This contradiction implies that the zero-point energy of a harmonic oscillator cannot be zero.

29. Show that for the $n = 0$ state of a harmonic oscillator whose classical amplitude of motion is A, $y = 1$ at $x = A$, where y is the quantity defined by Eq. (5.67).

【Sol】

When the classical amplitude of motion is A, the energy of the oscillator is

$$\frac{1}{2}kA^2 = \frac{1}{2}hn, \quad \text{so} \quad A = \sqrt{\frac{hn}{k}}.$$

Using this for x in Equation (5.67) gives

$$y = \sqrt{\frac{2pmn}{\hbar}}\sqrt{\frac{hn}{k}} = 2p\sqrt{\frac{mn^2}{k}} = 1,$$

where Equation (5.64) has been used to relate n, m and k.

31. Find the expectation values $\langle x \rangle$ and $\langle x^2 \rangle$ for the first two states of a harmonic oscillator.

【Sol】

The expectation values will be of the forms

$$\int_{-\infty}^{\infty} xy^*y\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} x^2y^*y\,dx$$

It is far more convenient to use the dimensionless variable y as defined in Equation (5.67). The necessary integrals will be proportional to

$$\int_{-\infty}^{\infty} ye^{-y^2}dy, \quad \int_{-\infty}^{\infty} y^2e^{-y^2}dy, \quad \int_{-\infty}^{\infty} y^3e^{-y^2}dy, \quad \int_{-\infty}^{\infty} y^4e^{-y^2}dy,$$

The first and third integrals are seen to be zero (see Example 5.7). The other two integrals may be found from tables, from symbolic-manipulation programs, or by any of the methods outlined at the end of this chapter or in Special Integrals for Harmonic Oscillators, preceding the solutions for Section 5.8 problems in this manual. The integrals are

$$\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{1}{2}\sqrt{p}, \qquad \int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3}{4}\sqrt{p}.$$

An immediate result is that $\langle x \rangle = 0$ for the first two states of any harmonic oscillator, and in fact $\langle x \rangle = 0$ for any state of a harmonic oscillator (if $x = 0$ is the minimum of potential energy). A generalization of the above to any case where the potential energy is a symmetric function of x, which gives rise to wave functions that are either symmetric or antisymmetric, leads to $\langle x \rangle = 0$. To find $\langle x^2 \rangle$ for the first two states, the necessary integrals are

$$\int_{-\infty}^{\infty} x^2 y_o{}^* y_0 dx = \left(\frac{2mn}{\hbar}\right)^{1/2} \left(\frac{\hbar}{2pmn}\right)^{3/2} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy$$

$$= \frac{\hbar}{2p^{3/2}mn}\frac{\sqrt{p}}{2} = \frac{(1/2)hn}{4p^2 mn^2} = \frac{E_0}{k};$$

$$\int_{-\infty}^{\infty} x^2 y_1{}^* y_1 dx = \left(\frac{2mn}{\hbar}\right)^{1/2} \left(\frac{\hbar}{2pmn}\right)^{3/2} \int_{-\infty}^{\infty} 2y^4 e^{-y^2} dy$$

$$= \frac{\hbar}{2p^{3/2}mn} 2\frac{3\sqrt{p}}{2} = \frac{(3/2)hn}{4p^2 mn^2} = \frac{E_1}{k}.$$

In both of the above integrals,

$$dx = \frac{dx}{dy}dy = \sqrt{\frac{\hbar}{2pmn}}dy$$

has been used, as well as Table 5.2 and Equation (5.64).

33. A pendulum with a 1.00-g bob has a massless string 250 mm long. The period of the pendulum is 1.00 s. (a) What is its zero-point energy? Would you expect the zero-point oscillations to be detectable? (b) The pendulum swings with a very small amplitude such that its bob rises a maximum of 1.00 mm above its equilibrium position. What is the corresponding quantum number?

【Sol】

(a) The zero-point energy would be

$$E_0 = \frac{1}{2}hn = \frac{h}{2T} = \frac{4.14\times10^{-15}\ \text{eV}\cdot\text{s}}{2(1.00\ \text{s})} = 2.07\times10^{-15}\ \text{eV},$$

which is not detectable.

(b) The total energy is $E = mgH$ (here, H is the maximum pendulum height, given as an uppercase letter to distinguish from Planck's constant), and solving Equation (5.70) for n,

$$n = \frac{E}{hn} - \frac{1}{2} = \frac{mgH}{h/T} = \frac{(1.00\times10^{-3}\ \text{kg})(9.80\ \text{m/s}^2)(1.00\ \text{s})}{6.63\times10^{-34}\ \text{J}\cdot\text{s}} - \frac{1}{2} = 1.48\times10^{28}.$$

37. Consider a beam of particles of kinetic energy E incident on a potential step at $x = 0$ that is U high, where $E > U$ (Fig. 5.19). (a) Explain why the solution $De^{-ik'x}$ (in the notation of appendix) has no physical meaning in this situation, so that $D = 0$. (b) Show that the transmission probability here is $T = CC^*v'/AA^*v_1 = 4k_1^2/(k_1 + k')^2$. (c) A 1.00-mA beam of electrons moving at $2.00\text{x}10^6$ m/s enters a region with a sharply defined boundary in which the electron speeds are reduced to $1.00\text{x}10^6$ m/s by a difference in potential. Find the transmitted and reflected currents.

【Sol】

(a) In the notation of the Appendix, the wave function in the two regions has the form

$$y_I = Ae^{ik_1x} + Be^{-ik_1x}, \quad y_{II} = Ce^{ik'x} + De^{-ik'x},$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar}}, \quad k' = \sqrt{\frac{2m(E-U)}{\hbar}}.$$

The terms corresponding to $\exp(ik_1x)$ and $\exp(ik'x)$ represent particles traveling to the left; this is possible in region I, due to reflection at the step at $x = 0$, but not in region II (the reasoning is the same as that which lead to setting $G = 0$ in Equation (5.82)). Therefore, the $\exp(-ik'x)$ term is not physically meaningful, and $D = 0$.

(b) The boundary condition at x= 0 are then

$$A + B = C, \quad ik_1A - ik_1B = ik'C \quad \text{or} \quad A - B = \frac{k'}{k_1}C.$$

Adding to eliminate B, $2A = \left(1 + \dfrac{k'}{k_1}\right)C$, so

$$\frac{C}{A} = \frac{2k_1}{k_1 + k'}, \quad \text{and} \quad \frac{CC^*}{AA^*} = \frac{4k_1^2}{(k_1 + k')^2}.$$

(c) The particle speeds are different in the two regions, so Equation (5.83) becomes

$$T = \frac{|y_{II}|^2 v'}{|y_I|^2 v_1} = \frac{CC^*}{AA^*}\frac{k'}{k_1} = \frac{4k_1k'}{(k_1 + k')^2} = \frac{4(k_1/k')}{((k_1/k') + 1)^2}.$$

For the given situation, $k_1/k' = v_1/v' = 2.00$, so T = (4x2)/(2+1)2 = 8/9. The transmitted current is (T)(1.00 mA) = 0.889 mA, and the reflected current is 0.111mA.

As a check on the last result, note that the ratio of the reflected current to the incident current is, in the notation of the Appendix,

$$R = \frac{|y_{I-}|^2 v_1}{|y_{I+}|^2 v_1} = \frac{BB^*}{AA^*}$$

Eliminating C from the equations obtained in part (b) from the continuity condition as x = 0,

$$A\left(1 - \frac{k'}{k_1}\right) = B\left(1 + \frac{k'}{k_1}\right), \quad \text{so} \quad R = \left(\frac{(k_1/k') - 1}{(k_1/k') + 1}\right) = \frac{1}{9} = 1 - T$$

1. Why is it natural that three quantum numbers are needed to describe an atomic electron (apart from electron spin)?

【sol】

Whether in Cartesian (x, y, z) or spherical coordinates, three quantities are needed to describe the variation of the wave function throughout space. The three quantum numbers needed to describe an atomic electron correspond to the variation in the radial direction, the variation in the azimuthal direction (the variation along the circumference of the classical orbit), and the variation with the polar direction (variation along the direction from the classical axis of rotation).

3. Show that

$$R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/2a_o}$$

is a solution of Eq. (6.14) and that it is normalized.

【sol】

For the given function,

$$\frac{d}{dr} R_{10} = -\frac{2}{a_0^{5/2}} e^{-r/a_o}, \quad \text{and}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{10}}{dr}\right)=-\frac{2}{a_o^{5/2}}\frac{1}{r^2}\left(2r-\frac{r^2}{a_o}\right)e^{-r/a_o}$$

$$=\left(\frac{1}{a_o^2}-\frac{2}{r\,a_o}\right)R_{10}$$

This is the solution to Equation (6.14) if l=0 (as indicated by the index of R_{10}),

$$\frac{2}{a_o}=\frac{2me^2}{h^2 4\pi\epsilon_o},\qquad \text{or}\qquad a_o=\frac{4\pi^2\epsilon_o h^2}{me^2}$$

which is the case, and

$$\frac{2m}{\hbar^2}E=-\frac{1}{a_o^2},\qquad \text{or}\qquad E=-\frac{e^2}{8\pi\epsilon_o a_o}=E_1$$

again as indicated by the index of R_{10}.

To show normalization,

$$\int_0^\infty |R_{10}|^2 r^2 dr=\frac{4}{a_o^3}\int_0^\infty r^2 e^{-2r/a_o}dr=\frac{1}{2}\int_0^\infty u^2 e^{-u}du,$$

where the substitution $u=2r/a_o$ has been made. The improper definite integral in u is known to have the value 2 and so the given function is normalized.

5. In Exercise 12 of Chap. 5 it was stated that an important property of the eigenfunctions of a system is that they are orthogonal to one another, which means that

$$\int_{-\infty}^{\infty} y_n^* y_m \, dV = 0 \qquad n \neq m$$

Verify that this is true for the azimuthal wave functions Φ_{m_l} of the hydrogen atom by calculating

$$\int_0^{2p} \Phi_{m_l}^* \Phi_{m_l'} \, df \qquad \text{for} \quad m_l \neq m_l'$$

【sol】

From Equation (6.15) the integral, apart from the normalization constants, is

$$\int_0^{2p} \Phi_{m_l}^* \Phi_{m_l'} \, df = \int_0^{2p} e^{-im_l f} e^{im_l' f} \, df,$$

It is possible to express the integral in terms of real and imaginary parts, but it turns out to be more convenient to do the integral directly in terms of complex exponentials:

$$\int_0^{2p} e^{-im_l f} e^{im_l' f} \, df = \int_0^{2p} e^{i(m_l' - m_l')f} \, df$$

$$= \frac{1}{i(m_l' - m_l)} \left[e^{i(m_l' - m_l')f} \right]_0^{2p} = 0$$

The above form for the integral is valid only for $m_l \neq m_l$' , which is given for this case. In evaluating the integral at the limits, the fact that $e^{i2pn} = 1$ for any integer n (in this case (m_l' – m_l)) has been used.

7. Compare the angular momentum of a ground-state electron in the Bohr model of the hydrogen atom with its value in the quantum theory.

【sol】

In the Bohr model, for the ground-state orbit of an electron in a hydrogen atom, $l = h/mv = 2\pi r$, and so $L = pr =$. In the quantum theory, zero-angular-momentum states (y spherically symmetric) are allowed, and $L = 0$ for a ground-state hydrogen atom.

9. Under what circumstances, if any, is L_z equal to L?

【sol】

From Equation (6.22), L_z must be an integer multiple of ; for L to be equal to L_z, the product $l(1+1)$, from Equation(6.21), must be the square of some integer less than or equal to l. But,

$$l^2 \leq l(l+1) < (l+1)^2$$

or any nonnegative l, with equality holding in the first relation only if $l = 0$. Therefore, $l(l + 1)$ is the square of an integer only if $l = 0$, in which case $L_z = 0$ and $L = L_z = 0$.

11. What are the possible values of the magnetic quantum number m_l of an atomic electron whose orbital quantum number is $l = 4$?

【sol】

From Equation (6.22), the possible values for the magnetic quantum number m_l are

$$m_l = 0, \pm 1, \pm 2, \pm 3, \pm 4,$$

a total of nine possible values.

13. Find the percentage difference between L and the maximum value of L_z for an atomic electron in p, d, and f states.

【sol】

The fractional difference between L and the largest value of L_z, is, for a given l,

$$\frac{L - L_{z,\max}}{L} = \frac{\sqrt{l(l+1)} - l}{\sqrt{l(l+1)}} = 1 - \sqrt{\frac{l}{l+1}}.$$

For a p state, $l = 1$ and $1 - \sqrt{\frac{1}{2}} = 0.29 = 29\%$

For a d state, $l = 2$ and $1 - \sqrt{\frac{2}{3}} = 0.18 = 18\%$

For a f state, $l = 3$ and $1 - \sqrt{\frac{3}{4}} = 0.13 = 13\%$

15. In Sec. 6.7 it is stated that the most probable value of r for a $1s$ electron in a hydrogen atom is the Bohr radius a_o. Verify this.

【sol】

Using $R_{10}(r)$ from Table 6.1 in Equation (6.25),

$$P(r) = \frac{4r^2}{a_o^3} e^{-2r/a_o}.$$

The most probable value of r is that for which $P(r)$ is a maximum. Differentiating the above expression for $P(r)$ with respect to r and setting the derivative equal to zero,

$$\frac{d}{dr}P(r) = \frac{4}{a_o^3}\left(2r - \frac{2r^2}{a_o}\right)e^{-2r/a_o} = 0, \quad \text{or}$$

$$r = \frac{r^2}{a_o} \quad \text{and} \quad r = 0, a_o$$

for an extreme. At $r = 0$, $P(r) = 0$, and because $P(r)$ is never negative, this must be a minimum. dp/dr ® 0 as r ® ∞, and this also corresponds to a minimum. The only maximum of $P(r)$ is at $r = a_o$, which is the radius of the first Bohr orbit.

17. Find the most probable value of r for a $3d$ electron in a hydrogen atom.

【sol】

Using $R_{20}(r)$ from Table 6.1 in Equation (6.25), and ignoring the leading constants (which would not affect the position of extremes),

$$P(r) = r^6 e^{-2r/3a_o}$$

The most probable value of r is that for which $P(r)$ is a maximum. Differentiating the above expression for $P(r)$ with respect to r and setting the derivative equal to zero,

$$\frac{d}{dr}P(r) = \left(6r^5 - \frac{2r^6}{3a_o}\right)e^{-2r/3a_o} = 0, \quad \text{or}$$

$$6r^5 = \frac{2r^6}{3a_o} \quad \text{and} \quad r = 0,\ 9a_o$$

for an extreme. At $r = 0$, $P(r) = 0$, and because $P(r)$ is never negative, this must be a minimum. dP/dr ® 0 as r ® ∞, and this also corresponds to a minimum. The only maximum of $P(r)$ is at r = $9a_o$, which is the radius of the third Bohr orbit.

19. How much more likely is the electron in a ground-state hydrogen atom to be at the distance a_o from the nucleus than at the distance $2a_o$?

【sol】

For the ground state, $n = 1$, the wave function is independent of angle, as seen from the functions $\Phi(f)$ and $\Theta(q)$ in Table 6.1, where for $n = 1$, $l = m_l$, = 0 (see Problem 6-14). The ratio of the probabilities is then the ratio of the product $r^2\,(R_{10}(r))^2$ evaluated at the different distances. Specially,

$$\frac{P(a_o)dr}{P(a_o/2)dr} = \frac{a_o^2 e^{-2a_o/a_o}}{(a_o/2)^2 e^{-2(a_o/2)/a_o}} = \frac{e^{-2}}{(1/4)e^{-1}} = \frac{4}{e} = 1.47$$

Similarly,

$$\frac{P(a_o)dr}{P(2a_o)dr} = \frac{a_o^2 e^{-2a_o/a_o}}{(2a_o)^2 e^{-2(2a_o)/a_o}} = \frac{e^{-2}}{(4)e^{-4}} = \frac{e^2}{4} = 1.85$$

21. The probability of finding an atomic electron whose radial wave function is $R(r)$ outside a sphere of radius r_o centered on the nucleus is

$$\int_{r_o}^{\infty} |R(r)|^2 r^2 dr$$

(a) Calculate the probability of finding a $1s$ electron in a hydrogen atom at a distance greater than a_o from the nucleus.
(b) When a $1s$ electron in a hydrogen atom is $2a_o$ from the nucleus, all its energy is potential energy. According to classical physics, the electron therefore cannot ever exceed the distance $2a_o$ from the nucleus. Find the probability $r > 2a_o$ for a $1s$ electron in a hydrogen atom.

【sol】
(a) Using $R_{10}(r)$ for the $1s$ radial function from Table 6.1,

$$\int_{a_o}^{\infty} |R(r)|^2 r^2 dr = \frac{4}{a_o^3}\int_{a_o}^{\infty} r^2 e^{-2r/a_o} dr = \frac{1}{2}\int_{2}^{\infty} u^2 e^{-u} du,$$

where the substitution $u = 2r/a_0$ has been made.
Using the method outlined at the end of this chapter to find the improper definite integral leads to

$$\frac{1}{2}\int_{2}^{\infty} u^2 e^{-u} du = \frac{1}{2}\left[-e^{-u}\left(u^2 + 2u + 2\right)\right]_2^{\infty} = \frac{1}{2}\left[e^{-2}10\right] = 0.68 = 68\%,$$

(b) Repeating the above calculation with 2 a_0 as the lower limit of the integral,

$$\frac{1}{2}\int_4^\infty u^2 e^{-u} du = \frac{1}{2}\left[-e^{-u}\left(u^2+2u+2\right)\right]_4^\infty = \frac{1}{2}\left[e^{-4}26\right] = 0.24 = 24\%,$$

23. Unsold's theorem states that for any value of the orbital quantum number l, the probability densities summed over all possible states from m_l = -1 to m_l = +1 yield a constant independent of angles q or f that is,

$$\sum_{m_l=-l}^{+l} |\Theta|^2 |\Phi|^2 = \text{constant}$$

This theorem means that every closed subshell atom or ion (Sec. 7.6) has a spherically symmetric distribution of electric charge. Verify Unsold's theorem for l = 0, l = 1, and l = 2 with the help of Table 6. 1.

【sol】For l = 0, only m_l = 0 is allowed, $\Phi(f)$ and $\Theta(q)$ are both constants (from Table 6.1)), and the theorem is verified.

For l = 1, the sum is

$$\frac{1}{2p}\frac{3}{4}\sin^2 q + \frac{1}{2p}\frac{3}{2}\cos^2 q + \frac{1}{2p}\frac{3}{4}\sin^2 q = \frac{3}{4p},$$

In the above, $\Phi^*\Phi = 1/2\pi$, which holds for any l and m_l, has been used. Note that one term appears twice, one for m_l = -1 and once for m_l = 1.
For l = 2, combining the identical terms for $m_l = \pm 2$ and $m_l = \pm 1$, and again using $\Phi^*\Phi = 1/2\pi$, the sum is

$$2\frac{1}{2\pi}\frac{15}{16}\sin^4\theta + 2\frac{1}{2\pi}\frac{15}{4}\sin^2\theta\cos^2\theta + \frac{1}{2\pi}\frac{10}{16}(3\cos^2\theta - 1)^2.$$

The above may he simplified by extracting the commons constant factors, to

$$\frac{5}{16\pi}[(3\cos^2\theta - 1)^2 + 12\sin^2\theta\cos^2\theta + 3\sin^4\theta].$$

Of the many ways of showing the term in brackets is indeed a constant, the one presented here, using a bit of hindsight, seems to be one of the more direct methods. Using the identity $\sin^2\theta = 1 - \cos^2\theta$ to eliminate $\sin\theta$,

$$(3\cos^2\theta - 1)^2 + 12\sin^2\theta\cos^2\theta + 3\sin^4\theta$$

$$= (9\cos^4\theta - 6\cos^2\theta + 1) + 12(1 - \cos^2\theta)\cos^2\theta + 3(1 - 2\cos^2\theta + \cos^4\theta) = 1,$$

and the theorem is verified.

25. With the help of the wave functions listed in Table 6.1 verify that $\Delta l = \pm 1$ for $n = 2 \rightarrow n = 1$ transitions in the hydrogen atom.

【sol】

In the integral of Equation (6.35), the radial integral will never. vanish, and only the angular functions $\Phi(f)$ and $\Theta(q)$ need to be considered. The $\Delta l = 0$ transition is seen to be forbidden, in that the product

$$(\Phi_0(f)\Theta_{00}(q))^*(\Phi_0(f)\Theta_{00}(q)) = \frac{1}{4p}$$

is spherically symmetric, and any integral of the form of Equation (6.35) must vanish, as the argument $u = x, y$ or z will assume positive and negative values with equal probability amplitudes. If $l = 1$ in the initial state, the integral in Equation (6.35) will be seen to to vanish if u is chosen appropriately. If $m_l = 0$ initially, and $u = z = r\cos q$ is used, the integral (apart from constants) is

$$\int_0^p \cos^2 q \sin q\, dq = \frac{2}{3} \neq 0$$

If $m_l = \pm 1$ initially, and $u = x = r\sin q \cos f$ is used, the q -integral is of the form

$$\int_0^p \sin^2 q\, dq = \frac{p}{2} \neq 0$$

and the f -integral is of the form

$$\int_0^{2p} e^{\pm i f} \cos f\, df = \int_0^{2p} \cos^2 f\, df = p \neq 0$$

and the transition is allowed.

27. Verify that the n = 3 ® n = 1 transition for the particle in a box of Sec. 5.8 is forbidden whereas the n = 3 ® n = 2 and n = 2 ® n = 1 transitions are allowed.

【sol】

The relevant integrals are of the form

$$\int_0^L x \sin\frac{npx}{L}\sin\frac{mpx}{L}\,dx.$$

The integrals may be found in a number of ways, including consulting tables or using symbolic-manipulation programs (see; for instance, the solution to Problem 5-15 for sample Maple commands that are easily adapted to this problem).

One way to find a general form for the integral is to use the identity

$$\sin a \sin b = \tfrac{1}{2}[\cos(a-b)-\cos(a+b)]$$

and the indefinite integral (found from integration by parts)

$$\int x\cos kx\,dx = \frac{x\sin kx}{k} - \frac{1}{k}\int \sin kx\,dx = \frac{x\sin kx}{k} + \frac{\cos kx}{k^2}$$

to find the above definite integral as

$$\frac{1}{2}\left[\frac{Lx}{(n-m)}\sin\frac{(n-m)px}{L} + \frac{L^2}{(n-m)^2p^2}\cos\frac{(n-m)px}{L} - \frac{Lx}{(n+m)p}\sin\frac{(n+m)px}{L} + \frac{L^2}{(n+m)^2p^2}\cos\frac{(n+m)px}{L}\right]_0^L,$$

where $n \neq m^2$ is assumed. The terms involving sines vanish, with the result of

$$\frac{L^2}{2p^2}\left[\frac{\cos(n-m)p-1}{(n-m)^2}-\frac{\cos(n+m)p-1}{(n+m)^2}\right].$$

If n and m axe both odd or both even, $n + m$ and $n - m$ are even, the arguments of the cosine terms in the above expression are even-integral multiples of p, and the integral vanishes. Thus, the $n = 3$ ® $n = 1$ transition is forbidden, while the $n = 3$ ® $n = 2$ and $n = 2$ ® $n = 1$ transitions are allowed.

To make use of symmetry arguments, consider that

$$\int_0^L\left(x-\frac{L}{2}\right)\sin\frac{npx}{L}\sin\frac{mpx}{L}dx=\int_0^L x\sin\frac{npx}{L}\sin\frac{mpx}{L}dx$$

for $n \neq m$, because the integral of L times the product of the wave functions is zero; the wave functions were shown to be orthogonal in Chapter 5 (again, see Problem 5-15). Letting $u=L/2-x$,

$$\sin\frac{npx}{L}=\sin\frac{np((L/2)-u)}{L}=\sin\left(\frac{np}{2}-\frac{npu}{L}\right)$$

This expression will be ± cos (npu/L) for n odd and ±sin (npu/L) for n even. The integrand is then an odd function of u when n and m are both even or both odd, and hence the integral is zero. If one of n or m is even and the other odd, the integrand is an even function of u and the integral is nonzero.

29. Show that the magnetic moment of an electron in a Bohr orbit of radius r_n is proportional to $\sqrt{r_n}$

【sol】

From Equation (6.39), the magnitude of the magnetic moment of an electron in a Bohr orbit is proportional to the magnitude of the angular momentum, and hence proportional to n. The orbital radius is proportional to n^2 (See Equation (4.13) or Problem 4-28), and so the magnetic moment is proportional to $\sqrt{r_n}$.

31. Find the minimum magnetic field needed for the Zeeman effect to be observed in a spectral line of 400-nm wavelength when a spectrometer whose resolution is 0.010 nm is used.

【sol】

See Example 6.4; solving for B,

$$B = \frac{\Delta l}{l^2}\frac{4pmc}{e} = \frac{0.010\times10^{-9}\text{ m}}{(400\times10^{-9}\text{ m})^2}\frac{4p(9.1\times10^{-31}\text{ kg})(3.0\times10^{8}\text{ m/s})}{(1.6\times10^{-19}\text{ C})} = 1.34\text{ T}$$

1. A beam of electrons enters a uniform 1.20-T magnetic field. (a) Find the energy difference between electrons whose spins are parallel and antiparallel to the field. (b) Find the wavelength of the radiation that can cause the electrons whose spins are parallel to the field to flip so that their spins are antiparallel.

【sol】

(a) Using Equations (7.4) and (6.41), the energy difference is,

$$\Delta E = 2m_{sz}B = 2m_B B = 2(5.79\times10^{-5}\ \text{eV/T})(1.20\ \text{T}) = 1.39\times10^{-4}\ \text{eV}$$

(b) The wavelength of the radiation that corresponds to this energy is

$$l = \frac{hc}{\Delta E} = \frac{1.24\times10^{-6}\ \text{eV}\cdot\text{m}}{1.39\times10^{-4}\ \text{eV}} = 8.93\,\text{mm}$$

Note that a more precise value of AB was needed in the intermediate calculation to avoid roundoff error.

3. Find the possible angles between the z axis and the direction of the spin angular-momentum vector **S.**

【sol】

For an electron, $s = (\sqrt{3}/2)\hbar,\ s_z = \pm(1/2)\hbar,$ and so the possible angles axe given by

$$\arccos\left(\frac{\pm(1/2)\hbar}{(\sqrt{3}/2)\hbar}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right) = 54.7^o,\ 125.3^o$$

5. Protons and neutrons, like electrons, are spin-½ particles. The nuclei of ordinary helium atoms, ${}^4_2\mathrm{He}$, contain two protons and two neutrons each; the nuclei of another type of helium atom, ${}^3_2\mathrm{He}$, contain two protons and one neutron each. The properties of liquid ${}^4_2\mathrm{He}$ and liquid ${}^3_2\mathrm{He}$ are different because one type of helium atom obeys the exclusion principle but the other does not. Which is which, and why?

【sol】

${}^4_2\mathrm{He}$ atoms contain even numbers of spin-½ particles, which pair off to give zero or integral spins for the atoms. Such atoms do not obey the exclusion principle. ${}^3_2\mathrm{He}$ atoms contain odd numbers of spin-½ particles, and so have net spins of $\frac{1}{2}$, $\frac{3}{2}$ or $\frac{5}{2}$, and they obey the exclusion principle.

7. In what way does the electron structure of an alkali metal atom differ from that of a halogen atom? From that of an inert gas atom?

【sol】

An alkali metal atom has one electron outside closed inner shells: A halogen atom lacks one electron of having a closed outer shell: An inert gas atom has a closed outer shell.

9. How many electrons can occupy an *f* subshell?

【sol】

For *f* subshell, with $l = 3$, the possible values of m_l are ±3, ±2, ±1 or 0, for a total of $2l + 1 = 7$ values of m_l. Each state can have two electrons of opposite spins, for a total of 14 electrons.

11. If atoms could contain electrons with principal quantum numbers up to and including $n = 6$, how many elements would there be?

【sol】

The number of elements would be the total number of electrons in all of the shells. Repeated use of Equation (7.14) gives

$$2n^2 + 2\,(n - 1)^2 + \ldots + 2\,(1)^2 = 2\,(36 + 25 + 16 + 9 + 4 + 1) = 182.$$

In general, using the expression for the sum of the squares of the first n integers, the number of elements would be

$$2\left(\tfrac{1}{6}n(2n+1)(n+1)\right) = \tfrac{1}{3}[n(2n+1)(n+1)],$$

which gives a total of 182 elements when $n = 6$.

13. The ionization energies of Li, Na, K, Rb, and Cs are, respectively, 5.4, 5.1, 4.3, 4.2, and 3.9 eV. All are in group 1 of the periodic table. Account for the decrease in ionization energy with increasing atomic number.

【sol】

All of the atoms are hydrogenlike, in that there is a completely filled subshell that screens the nuclear charge and causes the atom to "appear" to be a single charge. The outermost electron in each of these atoms is further from the nucleus for higher atomic number, and hence has a successively lower binding energy.

15. (a) Make a rough estimate of the effective nuclear charge that acts on each electron in the outer shell of the calcium (Z = 20) atom. Would you think that such an electron is relatively easy or relatively hard to detach from the atom? (b) Do the same for the sulfur (Z = 16) atom.

【sol】

(a) See Table 7.4. The $3d$ subshell is empty, and so the effective nuclear charge is roughly $+2e$, and the outer electron is relatively easy to detach.

(b) Again, see Table 7.4. The completely filled K and L shells shield $+10e$ of the nuclear charge of = $16e$; the filled $3s^2$ subshell will partially shield the nuclear charge, but not to the same extent as the filled shells, so $+6e$ is a rough estimate for the effective nuclear charge. This outer electron is then relatively hard to detach.

17. Why are Cl atoms more chemically active than Cl^- ions? Why are Na atoms more chemically active than Na^+ ions?

【sol】

Cl^- ions have closed shells, whereas a Cl atom is one electron short of having a closed shell and the relatively poorly shielded nuclear charge tends to attract an electron from another atom to fill the shell. Na^+ ions have closed shells, whereas an Na atom has a single outer electron that can be detached relatively easily in a chemical reaction with another atom.

19. In each of the following pairs of atoms, which would you expect to be larger in size? Why? Li and F; Li and Na; F and Cl; Na and Si.

【sol】

The Li atom ($Z = 3$) is larger because the effective nuclear charge acting on its outer electron is less than that acting on the outer electrons of the F atom ($Z = 9$). The Na atom ($Z = 11$) is larger because it has an additional electron shell (see Table 7.4). The Cl atom ($Z = 17$) atom is larger because has an additional electron shell. The Na atom is larger than the Si atom ($Z = 14$) for the same reason as given for the Li atom.

21. Why is the normal Zeeman effect observed only in atoms with an even number of electrons?

【sol】 The only way to produce a normal Zeeman effect is to have no net electron spin; because the electron spin is ±½, the total number of electrons must be even. If the total number of electrons were odd, the net spin would be nonzero, and the anomalous Zeeman effect would be observable.

23. The spin-orbit effect splits the $3P \rightarrow 3S$ transition in sodium (which gives rise to the yellow light of sodium-vapor highway lamps) into two lines, 589.0 nm corresponding to $3P_{3/2} \rightarrow 3S_{1/2}$ and 589.6 nm corresponding to $3P_{1/2} \rightarrow 3S_{1/2}$. Use these wavelengths to calculate the effective magnetic field experienced by the outer electron in the sodium atom as a result of its orbital motion.

【sol】

See Example 7.6. Expressing the difference in energy levels as

$$\Delta E = 2m_B B = hc\left(\frac{1}{l_1} - \frac{1}{l_2}\right); \qquad \text{solving for } B,$$

$$B = \frac{hc}{2m_B}\left(\frac{1}{l_1} - \frac{1}{l_2}\right)$$

$$= \frac{1.24 \times 10^{-6}\ \text{eV}\cdot\text{m}}{2 \times 5.79 \times 10^{-5}\ \text{eV/T}}\left(\frac{1}{589.0 \times 10^{-9}\ \text{m}} - \frac{1}{589.6 \times 10^{-9}\ \text{m}}\right) = 18.5\,\text{T}$$

25. If $j = \frac{5}{2}$, what values of l are possible?

【sol】

The possible values of l are $j + \frac{1}{2} = 3$ and $j - \frac{1}{2} = 2$.

27. What must be true of the subshells of an atom which has a $^1S_0$ ground state?

【sol】

For the ground state to be a singlet state with no net angular momentum, all of the subshells must be filled.

29. The lithium atom has one $2s$ electron outside a filled inner shell. Its ground state is $^2S_{1/2}$. (a) What are the term symbols of the other allowed states, if any? (b) Why would you think the $^2S_{1/2}$ state is the ground state?

【sol】

For this doublet state, L = 0, **S** = **J** = ½. There axe no other allowed states. This state has the lowest possible values of L and **J**, and is the only possible ground state.

31. The aluminum atom has two $3s$ electrons and one $3p$ electron outside filled inner shells. Find the term symbol of its ground state.

【sol】

The two $3s$ electrons have no orbital angular momentum, and their spins are aligned oppositely to give no net angular momentum. The $3p$ electron has $l = 1$, so L = 1, and in the ground state J = ½. The term symbol is $^2P_{1/2}$.

33. Why is it impossible for a $2^2D_{3/2}$ state to exist?

【sol】

A D state has L = 2; for a $2^2D_{3/2}$ state, $n = 2$ but L must always be strictly less than n, and so this state cannot exist.

35. Answer the questions of Exercise 34 for an f electron in an atom whose total angular momentum is provided by this electron.

【sol】

(a) From Equation (7.17), $j = l \pm \frac{1}{2} = \frac{5}{2}, \frac{7}{2}$.

(b) Also from Equation (7.17), the corresponding angular momenta are $\frac{\sqrt{35}}{2}\hbar$ and $\frac{\sqrt{63}}{2}\hbar$

(c) The values of L and S are $\sqrt{12}\hbar$ and $\frac{\sqrt{3}}{2}\hbar$. The law of cosines is

$$\cos q = \frac{J^2 - L^2 - S^2}{2LS},$$

where q is the angle between **L** and **S;** then the angles are,

$$\arccos\left(\frac{(35/4) - 12 - (3/4)}{2\sqrt{12}(\sqrt{3}/2)}\right) = \arccos\left(-\frac{2}{3}\right) = 132^o$$

and

$$\arccos\left(\frac{(63/4) - 12 - (3/4)}{2\sqrt{12}(\sqrt{3}/2)}\right) = \arccos\left(\frac{1}{2}\right) = 60.0^o$$

(d) The multiplicity is 2(1/2) + 1 = 2, the state is an f state because the total angular momentum is provided by the f electron, and so the terms symbols are $^2F_{5/2}$ and $^2F_{7/2}$.

37. The magnetic moment m_J of an atom in which LS coupling holds has the magnitude

$$m_J = \sqrt{J(J+1)}\,g_J m_B$$

where $m_B = e\hbar/2m$ is the Bohr magneton and

$$g_J = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

is the **Landé** g **factor.** (a) Derive this result with the help of the law of cosines starting from the fact that averaged over time, only the components of m_L and m_S parallel to **J** contribute to m_L **.** (b) Consider an atom that obeys LS coupling that is in a weak magnetic field **B** in which the coupling is preserved. How many substates are there for a given value of **J?** What is the energy difference between different substates?

【sol】

(a) In Figure 7.15, let the angle between **J** and **S** be *a* and the angle between **J** and **L** be *b*. Then, the product m_J has magnitude

$$2m_B|\mathrm{S}|\cos a + m_B|\mathrm{L}|\cos b = m_B|\mathrm{J}| + m_B|\mathrm{S}|\cos a = m_B|\mathrm{J}|\left(1+\frac{|\mathrm{S}|}{|\mathrm{J}|}\cos a\right)$$

In the above, the factor of 2 in $2m_B$ relating the electron spin magnetic moment to the Bohr magneton is from Equation (7.3). The middle term is obtained by using |S| cos *a* + |S| cos *b* = |J|**.** The above expression is equal to the product m_J because in this form, the magnitudes of the angular momenta include factors of h. From the law of cosines,

$$\cos a = \frac{|\mathrm{L}|^2 - |\mathrm{J}|^2 - |\mathrm{S}|^2}{-2|\mathrm{J}||\mathrm{S}|}$$

and so

$$\frac{|\mathrm{S}|}{|\mathrm{J}|}\cos a = \frac{|\mathrm{L}|^2 - |\mathrm{J}|^2 - |\mathrm{S}|^2}{2|\mathrm{J}|^2} = \frac{\mathbf{J}(\mathbf{J}+1) - \mathrm{L}(\mathrm{L}+1) + \mathbf{S}(\mathbf{S}+1)}{2\mathbf{J}(\mathbf{J}+1)}$$

and the expression for m_J in terms of the quantum numbers is

$$m_J \hbar = |J| g_J m_B, \quad \text{or} \quad m_J = \sqrt{J(J+1)} g_J m_B, \qquad \text{where}$$

$$g_J = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

(b) There will be one substate for each value of M_J, where $M_J = -J \ldots J$, for a total of $2J + 1$ substates. The difference in energy between the substates is

$$\Delta E = g_J m_B M_J B$$

39. Explain why the x-ray spectra of elements of nearby atomic numbers are qualitatively very similar, although the optical spectra of these elements may differ considerably.

【sol】

The transitions that give rise to x-ray spectra are the same in all elements since the transitions involve only inner, closed-shell electrons. Optical spectra, however, depend upon the possible states of the outermost electrons, which, together with the transitions permitted for them, are different for atoms of different atomic number.

41. Find the energy and the wavelength of the K_α x-rays of aluminum.

【sol】

From either of Equations (7.21) or (7.22),

$$E = (10.2 \text{ eV})\,(Z - 1)^2 = (10.2 \text{ eV})\,(144) = 1.47 \text{ keV}.$$

The wavelength is

$$l = \frac{hc}{E} = \frac{1.24\times10^{-6}\,\text{eV}\cdot\text{m}}{14.7\times10^{3}\,\text{eV}} = 8.44\times10^{-10}\text{ m} = 0.844\,\text{nm}$$

43. Distinguish between singlet and triplet states in atoms with two outer electrons.

【sol】

In a singlet state, the spins of the outer electrons are antiparrallel. In a triplet state, they are parallel

1. The energy needed to detach the electron from a hydrogen atom is 13.6 eV, but the energy needed to detach an electron from a hydrogen molecule is 15.7 eV. Why do you think the latter energy is greater?

【sol】

The nuclear charge of $+2e$ is concentrated at the nucleus, while the electron charges' densities are spread out in (presumably) the $1s$ subshell. This means that the additional attractive force of the two protons exceeds the mutual repulsion of the electrons to increase the binding energy.

3. At what temperature would the average kinetic energy of the molecules in a hydrogen sample be equal to their binding energy?

【sol】

Using 4.5 eV for the binding energy of hydrogen,

$$\frac{3}{2}kT = 4.5\,\text{eV} \quad \text{or} \quad T = \frac{2}{3}\frac{4.5\,\text{eV}}{8.62\times10^{-5}\ \text{eV/K}} = 3.5\times10^{4}\ \text{K}$$

5. When a molecule rotates, inertia causes its bonds to stretch. (This is why the earth bulges at the equator.) What effects does this stretching have on the rotational spectrum of the molecule?

【sol】

The increase in bond lengths in the molecule increases its moment of inertia and accordingly decreases the frequencies in its rotational spectrum (see Equation (8.9)). In addition, the higher the quantum number J (and hence the greater the angular momentum), the faster the rotation and the greater the distortion, so the spectral lines are no longer evenly spaced.

Quantitatively, the parameter I (the moment of inertia of the molecule) is a function of J, with I larger for higher J. Thus, all of the levels as given by Equation (8.11) are different, so that the spectral lines are not evenly spaced. (It should be noted that if I depends on J, the algebraic steps that lead to Equation (8.11) will not be valid.)

7. The J=0→J=1 rotational absorption line occurs at 1.153×10^{11} Hz in $^{12}C^{16}O$ and at 1.102×10^{11} Hz in $^{?}C^{16}O$. Find the mass number of the unknown carbon isotope.

【sol】

From Equation (8.11), the ratios of the frequencies will be the ratio of the moments of inertia. For the different isotopes, the atomic separation, which depends on the charges of the atoms, will be essentially the same. The ratio of the moments of inertia will then be the ratio of the reduced masses. Denoting the unknown mass number by x and the ratio of the frequencies as r, r in terms of x is

$$r = \frac{\frac{x \cdot 16}{x + 16}}{\frac{12 \cdot 16}{12 + 16}}$$

Solving for *x* in terms of *r*,

$$x = \frac{48r}{7 - 3r}$$

Using $r = (1.153)/(1.102)$ in the above expression gives $x = 13.007$, or the integer 13 to three significant figures.

9. The rotational spectrum of HCI contains the following wavelengths:

12.03×10^{-5} m, 9.60×10^{-5} m, 8.04×10^{-5} m, 6.89×10^{-5} m, 6.04×10^{-5} m

If the isotopes involved are ^{1}H and ^{35}Cl, find the distance between the hydrogen and chlorine nuclei in an HCl molecule.

【sol】

The corresponding frequencies are, from $n = c/l$, and keeping an extra significant figure, in multiples of 10^{12} Hz:

2.484, 3.113, 4.337, 4.947

The average spacing of these frequencies is Dv = 0.616 x 10^{12} Hz. (A least-squares fit from a spreadsheet program gives 0.6151 if c = 2.998 x 10^{8} m/s is used.) From Equation (8.11), the spacing of the frequencies should be Dv = /2pI ; Solving for I and using Dv as found above,

$$I = \frac{\hbar}{2p\Delta n} = \frac{1.055\times10^{-34}\,\mathrm{J\cdot s}}{2p(0.6151\times10^{12}\,\mathrm{Hz})} = 2.73\times10^{-47}\ \mathrm{kg\cdot m^2}$$

The reduced mass of the HCI molecule is (35/36)rn_{H}, and so the distance between the nuclei is

$$R = \sqrt{\frac{I}{m}} = \sqrt{\frac{36\times(2.73\times10^{-47}\ \mathrm{kg\cdot m^2})}{35\times(1.67\times10^{-27}\ \mathrm{kg})}} = 0.129\,\mathrm{nm}$$

(keeping extra significant figures in the intermediate calculation gives a result that is rounded to 0.130 nm to three significant figures).

11. A $^{200}Hg^{35}Cl$ Molecule emits a 4.4-cm photon when it undergoes a rotational transition from j = 1 to j = 0. Find the interatomic distance in this molecule.

【sol】

Using $n_{1\to 0} = c/l$ and $I = m' R^2$ in Equation (8.11) and solving for R,

$$R^2 = \frac{\hbar l}{2pm'c}$$

For this atom, $m' = m_H(200\text{x}35)/(200 + 35)$, and

$$R = \sqrt{\frac{(1.055\times 10^{-34}\,\text{J}\cdot\text{s})(4.4\times 10^{-2}\,\text{m})}{2p(1.67\times 10^{-27}\,\text{kg})(3.0\times 10^{8}\,\text{m/s})}} = 0.223\ \text{nm}$$

or 0.22 nm to two significant figures.

13. In Sec. 4.6 it was shown that, for large quantum numbers, the frequency of the radiation from a hydrogen atom that drops from an initial state of quantum number n to a final state of quantum number n - 1 is equal to the classical frequency of revolution of an electron in the n-th Bohr orbit. This is an example of Bohr's correspondence principle. Show that a similar correspondence holds for a diatomic molecule rotating about its center of mass.

【sol】

Equation (8.11) may be re-expressed in terms of the frequency of the emitted photon when the molecule drops from the J rotational level to the J - 1 rotational level,

$$n_{J \to J-1} = \frac{\hbar J}{2pI}.$$

For large J, the angular momentum of the molecule in its initial state is

$$L = \hbar\sqrt{J(J+1)} = \hbar J\sqrt{1+1/J} \approx \hbar J$$

Thus, for large J,

$$n \approx \frac{L}{2pI}, \quad \text{or} \quad L = wI,$$

the classical expression.

15. The hydrogen isotope deuterium has an atomic mass approximately twice that of ordinary hydrogen. Does H_2 or HD have the greater zero-point energy? How does this affect the binding energies of the two molecules?

【sol】

The shape of the curve in Figure 8.18 will be the same for either isotope; that is, the value of k in Equation (8.14) will be the same. HD has the greater reduced mass, and hence the smaller frequency of vibration v_o and the smaller zero- point energy. HD is the more tightly bound, and has the greater binding energy since its zero-point energy contributes less energy to the splitting of the molecule.

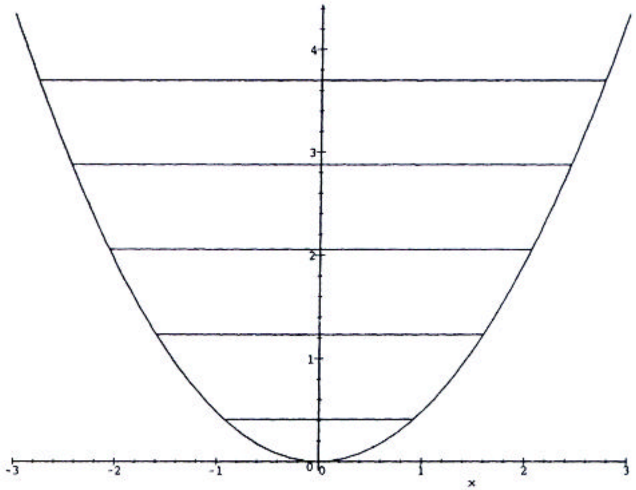
17. The force constant of the $^1H^{19}F$ molecule is approximately 966 N/m. (a) Find the frequency of vibration of the molecule. (b) The bond length in $^1H^{19}F$ is approximately 0.92 nm. Plot the potential energy of this molecule versus internuclear distance in the vicinity of 0.92 nm and show the vibrational energy levels as in Fig. 8.20.

【sol】

(a) Using $m' = (19/20)m_H$ in Equation (8.15),

$$n_o = \frac{1}{2p}\sqrt{\frac{966\ \text{N/m}}{1.67\times10^{-27}\ \text{kg}}\frac{20}{19}} = 1.24\times10^{14}\ \text{Hz}$$

(b) $E_o = \frac{1}{2}\hbar\sqrt{\frac{k}{m'}}$ = 4.11 X 10^{-20} J. The levels are shown below, where the vertical scale is in units of 10^{-20} J and the horizontal scale is in units of 10^{-11} m.



19. The lowest vibrational states of the $^{23}Na^{35}Cl$ molecule are 0.063 eV apart. Find the approximate force constant of this molecule.

【sol】

From Equation (8.16), the lower energy levels are separated by $\Delta E = h\nu_o$, and $\nu_o = \Delta E / h$. Solving Equation (8.15) for k,

$$k = m'(2\pi\nu_o)^2 = m'\left(\frac{\Delta E}{\hbar}\right)$$

Using $m' = m_H\,(23\cdot 35)/(23 + 35)$,

$$k = \frac{23\cdot 35}{58}(1.67\times 10^{-27}\,\text{kg})\left(\frac{(0.063\text{eV})(1.60\times 10^{-19}\ \text{J/eV})}{4.14\times 10^{-15}\ \text{eV}\cdot\text{s}}\right) = 213\ \text{N/m}$$

or 2.1×10^2 N/m to the given two significant figures.

21. The bond between the hydrogen and chlorine atoms in a $^{1}H^{35}Cl$ molecule has a force constant of 516 N/m. Is it likely that an HCl molecule will be vibrating in its first excited vibrational state at room temperature? Atomic masses are given in the Appendix.

【sol】 Using

$$\Delta E = h\nu_o = \hbar\sqrt{\frac{k}{m'}} \qquad \text{and} \qquad m' = m_H\frac{35}{36},$$

$$\Delta E = (1.055\times 10^{-34}\,\text{J}\cdot\text{s})\sqrt{\frac{516\ \text{N/m}}{1.67\times 10^{-27}\ \text{kg}}\frac{36}{35}} = 5.94\times 10^{-20}\ \text{J} = 0.371\,\text{eV}$$

At room temperature of about 300 K,

$$k\,T = (8.617 \times 10^{-5}\ \text{eV/K})\ (300\ \text{K}) = 0.026\ \text{eV}.$$

An individual atom is not likely to he vibrating in its first excited level, but in a large collection of atoms, it is likely that some of these atoms will be in the first excited state. It's important to note that in the above calculations, the symbol "k" has been used for both a spring constant and Boltzmann's constant, quantities that are not interchangeable.

1. At what temperature would one in a thousand of the atoms in a gas of atomic hydrogen be in the n=2 energy level?

【sol】

$$g(e_2) = 8, \quad g(e_1) = 2$$

Then, $$\frac{n(e_2)}{n(e_1)} = \frac{1}{1000} = 4e^{-(e_2 - e_1)/kT} = 4e^{3e_1/kT}$$

$$T = \left(\frac{1}{k}\right)\frac{(3/4)(-e_1)}{\ln 4000} = \frac{(3/4)(13.6\,\text{eV})}{(8.62\times10^{-5}\ \text{eV/K})(\ln 4000)} = 1.43\times10^{4}\ \text{K}$$

where $e_2 = e_1/4$, and $e_1 = -13.6$ eV

3. The $3^2P_{1/2}$ first excited sate in sodium is 2.093 eV above the $3^2S_{1/2}$ ground state. Find the ratio between the numbers of atoms in each state in sodium vapor at l200 K. (see Example 7.6.)

【sol】

multiplicity of P-level : $2L+1=3$, multiplicity of S-level : 1

The ratio of the numbers of atoms in the states is then,

$$\left(\frac{3}{1}\right)\exp\left(-\frac{2.09\ \text{eV}}{(8.62\times10^{-5}\ \text{eV/K})(1200\ \text{K})}\right) = 4.86\times10^{-9}$$

5. The moment of inertia of the H_2 molecule is 4.64×10^{-48} kg· m². (a) Find the relative populations of the J=0,1,2,3, and 4 rotational states at 300 K. (b) can the populations of the J=2 and J=3 states ever be equal? If so, at what temperature does this occur?

【sol】

(a) $g(J) = 2J + 1, \quad e_J = \dfrac{J(J+1)\hbar^2}{2I} \qquad e_{J=0} = 0$

$$\frac{N(J)}{N(J=0)} = (2J+1)\exp\left(-\frac{J(J+1)\hbar^2}{2IkT}\right) = (2J+1)\left[\exp\left(-\frac{\hbar^2}{2IkT}\right)\right]^{J(J+1)}$$

$$= (2J+1)\left[\exp\left(-\frac{(1.06\times10^{-34}\ \mathrm{J\cdot s})^2}{2(4.64\times10^{-48}\ \mathrm{kg\cdot m^2})(1.38\times10^{-23}\ \mathrm{J/K})(300\ \mathrm{K})}\right)\right]^{J(J+1)}$$

$$= (2J+1)[0.749]^{J(J+1)}$$

Applying this expression to J=0, 1, 2, 3, and 4 gives, respectively, 1 exactly, 1.68, 0.880, 0.217, and 0.0275.

(b) Introduce the dimensionless parameter . Then, for the populations of the J=2 and J=3 states to be equal,

$$5x^6 = 7x^{12}, \quad x^6 = \frac{5}{7} \quad \text{and} \quad 6\ln x = \ln\frac{5}{7}$$

Using , $\ln x = -\hbar^2 / 2IkT$ and $\ln(5/7) = -\ln(7/5)$ and solving for T,

$$T = \frac{6\hbar^2}{2Ik\ln(7/5)}$$

$$= \frac{6(1.05\times10^{-34}\ \mathrm{J\cdot s})^2}{2(4.64\times10^{-48}\ \mathrm{kg\cdot m^2})(1.38\times10^{-23}\ \mathrm{J/K})\ln(1.4)} = 1.55\times10^3\mathrm{K}$$

7. Find $\bar{v}$ and v_{rms} for an assembly of two molecules, one with a speed of 1.00 m/s and the other with a speed of 3.00 m/s.

【sol】

$$\bar{v} = \tfrac{1}{2}(1.00 + 3.00) = 2.00\ \ (\mathrm{m/s})$$

$$v_{rms} = \sqrt{\tfrac{1}{2}[1.00^2 + 3.00^2]} = 2.24\ \ (\mathrm{m/s})$$

9. At what temperature will the average molecular kinetic energy in gaseous hydrogen equal the binding energy of a hydrogen atom?

【sol】

For a monatomic hydrogen, the kinetic energy is all translational and $\overline{KE} = \tfrac{3}{2}kT$

solving for T with $\overline{KE} = -E_1$

$$T = \frac{2}{3}\left(-\frac{E_1}{k}\right) = \frac{(2/3)(13.6\ \mathrm{eV})}{(8.62\times10^{-5}\ \mathrm{eV/K})} = 1.05\times10^5\ \mathrm{K}$$

11. Find the width due to the Doppler effect of the 656.3-nm spectral line emitted by a gas of atomic hydrogen at 500 K.

【sol】

For nonrelativistic atoms, the shift in wavelength will be between $+l(v/c)$ and $-l(v/c)$ and the width of the Doppler-broadened line will be $2l(v/c)$. Using the rms speed from KE=$(3/2)kT = (1/2)mv^2$, $v = \sqrt{3kT/m}$, and

$$\Delta l = 2l\frac{\sqrt{3kT/m}}{c}$$

$$= 2(656.3\times10^{-9}\,\text{m})\frac{\sqrt{3(1.38\times10^{-23}\,\text{J/K})(500\,\text{K})/(1.67\times10^{-27}\,\text{kg})}}{3.0\times10^{8}\,\text{m/s}}$$

$$= 1.54\times10^{-11}\,\text{m} = 15.4\,\text{pm}$$

13. Verify that the average value of $1/v$ for an ideal-gas molecule is $\sqrt{2m/pkT}$.

[Note : $\int_0^\infty ve^{-av^2}dv = 1/(2a)$]

【sol】

The average value of $1/v$ is $\left\langle\frac{1}{v}\right\rangle = \frac{1}{N}\int_0^\infty \frac{1}{v}n(v)dv$

$$= \frac{1}{N}4pN\left(\frac{m}{2pkT}\right)^{3/2}\int_0^\infty ve^{-mv^2/2kT}dv$$

$$= 4p\left(\frac{m}{2pkT}\right)^{3/2}\left(\frac{kT}{m}\right) = \sqrt{\frac{2m}{pkT}} = 2\frac{1}{<v>}$$

17. How many independent standing waves with wavelengths between 95 and 10.5 mm can occur in a cubical cavity 1 m on a side? How many with wavelengths between 99.5 and 100.5 mm? (Hint: First show that $g(\lambda)d\lambda = 8\pi L^3\, d\lambda/\lambda^4$.)

【sol】

The number of standing waves in the cavity is

$$g(\nu)d\nu = \frac{8\pi L^3 \nu^2}{c^3}d\nu$$

$$g(\lambda)d\lambda = g(\nu)d\nu = \frac{8\pi L^3}{c^3}\left(\frac{c}{\lambda}\right)^2\frac{c}{\lambda^2}d\lambda = \frac{8\pi L^3}{\lambda^4}d\lambda$$

Therefore the number of standing waves between 9.5mm and 10.5mm is

$$g(\lambda)d\lambda = \frac{8\pi(1\,\text{m})^3}{(10\,\text{mm})^4}(1.0\,\text{mm}) = 2.5\times10^6$$

Similarly, the number of waves between99.5mm and 100.5mm is $2.5\text{x}10^2$, lower by a factor of 10^4.

l9. A thermograph measures the rate at which each small portion of a persons skin emits infrared radiation. To verify that a small difference in skin temperature means a significant difference in radiation rate, find the percentage difference between the total radiation from skin at 34^o and at 35^oC.

【sol】

By the Stefan-Boltzmann law, the total energy density is proportional to the fourth power of the absolute temperature of the cavity walls, as

$$R = sT^4$$

The percentage difference is

$$\frac{sT_1^4 - sT_2^4}{sT_1^4} = \frac{T_1^4 - T_2^4}{T_1^4} = 1 - \left(\frac{T_2}{T_1}\right)^4 = 1 - \left(\frac{307\text{ K}}{308\text{ K}}\right) = 0.013 = 1.3\%$$

For temperature variations this small, the fractional variation may be approximated by

$$\frac{\Delta R}{R} = \frac{\Delta(T^4)}{T^4} = \frac{3T^3\Delta T}{T^4} = 3\frac{\Delta T}{T} = 3\frac{1\text{K}}{308\text{K}} = 0.013$$

21. At what rate would solar energy arrive at the earth if the solar surface had a temperature 10 percent lower than it is?

【sol】

Lowering the Kelvin temperature by a given fraction will lower the radiation by a factor equal to the fourth power of the ratio of the temperatures. Using 1.4 kW/m^2 as the rate at which the sun's energy arrives at the surface of the earth

$$(1.4\text{ kW/m}^2)(0.90)^4 = 0.92\text{ kW/m}^2\ (= 66\%)$$

23. An object is at a temperature of 400°C. At what temperature would it radiate energy twice as fast?

【sol】

To radiate at twice the radiate, the fourth power of the Kelvin temperature would need to double.

Thus, $2[(400+273)\,\text{K}]^4 = T^4 \qquad T = 673\times 2^{1/4}\,\text{K} = 800\,\text{K}(527^o\text{C})$

25. At what rate does radiation escape from a hole l0 cm^2 in area in the wall of a furnace whose interior is at 700°C?

【sol】

The power radiated per unit area with unit emissivity in the wall is $P = sT^4$. Then the power radiated for the hole in the wall is

$$P' = sT^4 A = (5.67\times10^{-8}\,\text{W/(m}^2\cdot\text{K}^4))(973\,\text{K})^4(10\times10^{-4}\,\text{m}^2) = 51\,\text{W}$$

27. Find the surface area of a blackbody that radiates 100 kW when its temperature is 500°C. If the blackbody is a sphere, what is its radius?

【sol】

The radiated power of the blackbody (assuming unit emissivity) is

$$P = AesT^4 \qquad A = \frac{P}{esT^4} = \frac{100\times10^3\,\text{W}}{(1)(5.67\times10^{-8}\,\text{W/(m}^2\cdot\text{K}^4))((500+273)\text{K})^4}$$

$$= 4.94\times10^{-2}\,\text{m}^2 = 494\,\text{cm}^2$$

The radius of a sphere with this surface area is, then,

$$A = 4\pi r^2 \qquad r = \sqrt{A / 4\pi} = 6.27 \text{ cm}$$

31. The brightest part of the spectrum of the star Sirius is located at a wavelength of about 290 nm. What is the surface temperature of Sirius?

【sol】

From the Wien's displacement law, the surface temperature of Sirius is

$$T = \frac{2.898\times 10^{-3}\ \text{m}\cdot\text{K}}{\lambda_{max}} = \frac{2.898\times 10^{-3}\ \text{m}\cdot\text{K}}{290\times 10^{-9}\ \text{m}} = 1.0\times 10^{4}\ \text{K}$$

33. A gas cloud in our galaxy emits radiation at a rate of $1.0\text{x}10^{27}$ W. The radiation has its maximum intensity at a wavelength of 10 μm. If the cloud is spherical and radiates like a blackbody, find its surface temperature and its diameter.

【sol】

From the Wien's displacement law, the surface temperature of cloud is

$$T = \frac{2.898\times 10^{-3}\ \text{m}\cdot\text{K}}{10\times 10^{-6}\ \text{m}} = 2.9\times 10^{2}\ \text{K} = 290\ \text{K} = 17^{\circ}\text{C}$$

Assuming unit emissivity, the radiation rate is $R = \sigma T^4 = \frac{P}{A} = \frac{P}{\pi D^2}$

where D is the cloud's diameter. Solving for D,

$$D = \sqrt{\frac{P}{\pi\sigma T^4}} = \left(\frac{1.0\times 10^{27}\ \text{W}}{\pi(5.67\times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(290\ \text{K})^4}\right)^{1/2} = 8.9\times 10^{11}\text{m}$$

35. Find the specific heat at constant volume of 1.00 cm^3 of radiation in thermal equilibrium at 1000 K.

【sol】

The total energy(U) is related to the energy density by $U=Vu$, where V is the volume.

In terms of temperature,

$$U = Vu = VaT^4 = V\frac{4s}{c}T^4$$

The specific heat at constant volume is then

$$c_V = \frac{\partial U}{\partial T} = \frac{16s}{c}T^3V$$

$$= \frac{16(5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4)}{2.998\times10^8\ \text{m/s}}(1000\text{K})^3(1.0\times10^{-6}\,\text{m}^3)$$

$$= 3.03\times10^{-12}\ \text{J/K}$$

37. Show that the median energy in a free-electron gas at T=0 is equal to $e_F/2^{2/3}=0.630e_F$.

【sol】

At T=0, all states with energy less than the Fermi energy e_F are occupied, and all states with energy above the Fermi energy are empty. For $0\le e\le e_F$, the electron energy distribution is proportional to $\sqrt{e}$. The median energy is that energy for which there are many occupied states below the median as there are above. The median energy e_M is then the energy such that

$$\int_0^{e_M}\sqrt{e}\,de = \frac{1}{2}\int_0^{e_F}\sqrt{e}\,de$$

Evaluating the integrals,

$$\tfrac{2}{3}(e_M)^{3/2} = \tfrac{1}{3}(e_F)^{3/2}, \quad \text{or} \quad e_{\mathrm{M}} = (\tfrac{1}{2})^{3/2} e_F = 0.63 e_F$$

39. The Fermi energy in silver is 5.51 eV. (a)What is the average energy of the free electrons in silver at O K? (b)What temperature is necessary for the average molecular energy in an ideal gas to have this value? (c)What is the speed of an electron with this energy?

【sol】

(a) The average energy at T=0 K is $\overline{e_0} = \tfrac{3}{5} e_F = 3.31\ \mathrm{eV}$

(b) Setting $(3/2)kT=(3/5)e_{\mathrm{F}}$ and solving for T,

$$T = \frac{2}{5}\frac{e_F}{k} = \frac{2}{5}\frac{5.51\,\mathrm{eV}}{8.62\times 10^{-5}\ \mathrm{eV/K}} = 2.56\times 10^{4}\ \mathrm{K}$$

(c) The speed in terms of the kinetic energy is

$$v = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{6e_F}{5m}} = \sqrt{\frac{6(5.51\ \mathrm{eV})(1.602\times 10^{-19}\ \mathrm{J/eV})}{5(9.11\times 10^{-31}\ \mathrm{kg})}} = 1.08\times 10^{6}\ \mathrm{m/s}$$

43. Show that, if the average occupancy of a state of energy $e_{\mathrm{F}}+\Delta e$ is f_1 at any temperature, then the average occupancy of a state of energy $e_{\mathrm{F}}-\Delta e$ is $f_2=1-f_1$. (This is the reason for the symmetry of the curves in Fig.9.10 about e_{F}.)

【sol】

Using the Fermi-Dirac distribution function

$$f_1 = f_{FD}(e_F + \Delta e) = \frac{1}{e^{\Delta e/kT} + 1} \qquad f_2 = f_{FD}(e_F - \Delta e) = \frac{1}{e^{-\Delta e/kT} + 1}$$

$$f_1 + f_2 = \frac{1}{e^{\Delta e/kT} + 1} + \frac{1}{e^{-\Delta e/kT} + 1} = \frac{1}{e^{\Delta e/kT} + 1} + \frac{e^{\Delta e/kT}}{e^{\Delta e/kT} + 1} = 1$$

45. The density of zinc is 7.13 g/cm^3 and its atomic mass is 65.4 u. The electronic structure of zinc is given in Table 7.4, and the effective mass of an electron in zinc is 0.85 m_e. Calculate the Fermi energy in zinc.

【sol】

Zinc in its ground state has two electrons in $4s$ subshell and completely filled K, L, and M shells. Thus, there are two free electrons per atom. The number of atoms per unit volume is the ratio of the mass density r_{Zn} to the mass per atom m_{Zn}. Then,

$$e_F = \frac{h^2}{2m^*}\left(\frac{3(2)\, r_{Zn}}{8p\, m_{Zn}}\right)^{2/3}$$

$$= \left(\frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(0.85)(9.11\times10^{-31}\ \text{kg})}\right)\left(\frac{3(2)(7.13\times10^{3}\ \text{kg/m}^3)}{8p\,(65.4\text{u})(1.66\times10^{-27}\ \text{kg/u})}\right)^{2/3}$$

$$= 1.78\times10^{-18}\ \text{J} = 11\ \text{eV}$$

47. Find the number of electron states per electronvolt at $e=e_F/2$ in a 1.00-g sample of copper at O K. Are we justified in considering the electron energy distribution as continuous in a metal?

【sol】

At T=0, the electron distribution $n(e)$ is $n(e)=\frac{3N}{2}(e_F)^{-3/2}\sqrt{e}$

At $e=e_F/2$, $n\left(\frac{e_F}{2}\right)=\frac{3}{\sqrt{8}}\frac{N}{e_F}$

The number of atoms is the mass divided by the mass per atom,

$$N=\frac{(1.00\times10^{-3}\text{ kg})}{(63.55\text{ u})(1.66\times10^{-27}\text{ kg/u})}=9.48\times10^{21}$$

with the atomic mass of copper from the front endpapers and e_F=7.04 eV. The number of states per electronvolt is

$$n\left(\frac{e_F}{2}\right)=\frac{3}{\sqrt{8}}\frac{9.48\times10^{21}}{7.04\text{ eV}}=1.43\times10^{21}\text{ states/eV}$$

and the distribution may certainly be considered to be continuous.

49. The Bose-Einstein and Fermi-Dirac distribution functions both reduce to the Maxwell-Boltzmann function when $e^{a}e^{e/kT}$>>1. For energies in the neighborhood of kT, this approximation holds if e^{α}>>1. Helium atoms have spin 0 and so obey Bose-Einstein statistics verify that $f(e)=1/e^{\alpha}e^{e/kT}\approx Ae^{-e/kT}$ is valid for He at STP (20°C and atmospheric pressure, when the volume of 1 kmol of any gas is=22.4 m^3) by showing that *of* A<< l under these circumstances. To do this, use Eq(9.55) for $g(e)de$ with a coefficient of 4 instead of 8 since a He atom does not have the two spin states of an electron, and employing the approximation, find A from the normalization condition $n(e)de=N$, where N is the total number of atoms in the sample. (A kilomole of He contains Avogadro' s number No atoms, the atomic mass of He is 4.00 u and

$$\int_0^{\infty}\sqrt{x}e^{-ax}dx = \sqrt{p/a}/2a$$

【sol】

Using the approximation $f(e)=Ae^{-e/kT}$, and a factor of 4 instead of 8 in Equation (9.55), Equation (9.57) becomes

$$n(e)de = g(e)f(e)de = A4\sqrt{2}p\frac{Vm^{3/2}}{h^3}\sqrt{e}e^{-e/kT}de$$

Integrating over all energies,

$$N = \int_0^{\infty} n(e)de = A4\sqrt{2}p\frac{Vm^{3/2}}{h^3}\int_0^{\infty}\sqrt{e}e^{-e/kT}de$$

·The integral is that given in the problem with $x = \varepsilon$ and $a = kT$,

$$\int_0^{\infty} \sqrt{e}\, e^{-e/kT}\, de = \frac{\sqrt{p(kT)^3}}{2}$$

, so that

$$N = A4\sqrt{2}p \frac{Vm^{3/2}}{h^3} \frac{\sqrt{p(kT)^3}}{2} = A \frac{V}{h^3}(2pmkT)^{3/2}$$

Solving for A,

$$A = \frac{N}{V} h^3 (2pmkT)^{-3/2}$$

Using the given numerical values,

$$A = \frac{6.022\times10^{26}\ \text{kmol}^{-1}}{22.4\ \text{kg/kmol}}(6.626\times10^{-34}\ \text{J}\cdot\text{s})^3 \times [2p(4.00\,\text{u})(1.66\times10^{-27}\ \text{kg/u})(1.38\,1\times10^{-23}\ \text{J/K})(293\text{K})]^{-3/2}$$

$$= 3.56\times10^{-6},$$

which is much less than one.

51. The Fermi-Dirac distribution function for the free electrons in a metal cannot be approximated by the Maxwell-Boltzmann function at STP for energies in the neighborhood of kT. Verify this by using the method of Exercise 49 to show that $A>1$ in copper if $f(\varepsilon)\approx A\exp(\varepsilon/kT)$. As calculated in Sec. 9.9 N/V=8.48x10^{28} electrons/m^3 for copper. Note that Eq.(9.55) must be used unchanged here.

【sol】

Here, the original factor of 8 must be retained, with the result that

$$A=\frac{1}{2}\frac{N}{V}h^3(2\pi m_e kT)^{-3/2}$$

$$=\tfrac{1}{2}(8.48\times10^{26}\ \text{m}^{-3})(6.63\times10^{-34}\ \text{J}\cdot\text{s})^3\times[2\pi(9.11\times10^{-31})(1.38\times10^{-23}\ \text{J/K})(293\ \text{K})]^{-3/2}$$

$$=3.50\times10^3,$$

Which is much greater than one, and so the Fermi-Dirac distribution cannot be approximated by a Maxwell-Boltzmann distribution.